## Beyond Worst-case Analysis of Multicore Caching Strategies

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#### Abstract

Every processor with multiple cores sharing a cache needs to implement a cache-replacement algorithm. Previous work demonstrated that the competitive ratio of a large class of online algorithms, including Least-Recently-Used (LRU), grows with the length of the input. Furthermore, even offline algorithms like Furthest-In-Future, the optimal algorithm in single-core caching, cannot compete in the multicore setting. These negative results motivate a more in-depth comparison of multicore caching algorithms via alternative analysis measures. Specifically, the power of the adversary to adapt to online algorithms suggests the need for a direct comparison of online algorithms to each other.

In this paper, we introduce cyclic analysis, a generalization of bijective analysis introduced by Angelopoulos and Schweitzer [JACM'13]. Cyclic analysis captures the advantages of bijective analysis while offering flexibility that makes it more useful for comparing algorithms for a variety online problems. In particular, we take the first steps beyond worst-case analysis for analysis of multicore caching algorithms. We use cyclic analysis to establish relationships between multicore caching algorithms, including the advantage of LRU over all other multicore caching algorithms in the presence of locality of reference.

#### 1 Introduction

Despite the widespread use of multiple cores in a single machine, the theoretical performance of even the most common cache eviction algorithms is not yet fully understood when multiple cores simultaneously share a cache. Caching algorithms for multicore architectures have been well-studied in practice, including dynamic cache-partitioning heuristics [34, 38, 40] and operating system cache management [33, 41, 22]. There are very few theoretical guarantees, however, for performance of these algorithms. Furthermore, most existing guarantees on online multicore caching algorithms are negative [31, 25], but resource augmentation may be helpful in some cases [1, 2].

In this paper, we explore the *multicore caching*<sup>1</sup> **problem** in which multiple cores share a cache and request pages in an online manner. Upon serving a request, the requested page should become available in the shared cache. If the page is already in the cache, a hit takes place; otherwise, when the page is not in the cache, the core that issues the request incurs a miss. In case of a miss, the requested page should be fetched to the cache from a slow memory. Fetching a page causes a *fetch delay* in serving the subsequent requests made by the core that incurs the miss. Such delay is captured by the free-interleaving model of multicore caching [31, 27]. Under this model, when a core incurs a miss, it spends multiple cycles fetching the page from the slow memory while other cores may continue serving their requests in the meantime. Therefore, an algorithm's eviction strategy not only defines the state of the cache and the number of misses, but also the order in which requests are served. That is, a caching algorithm implicitly defines a "schedule" of requests served at each timestep through its previous eviction decisions.

Divergence between multicore and singlecore caching. Previous work [31, 25] leveraged the scheduling aspect of multicore caching to demonstrate that guarantees on competitive ratio<sup>2</sup> of algorithms in the single-core setting do not extend to multicore caching. In particular, López-Ortiz and Salinger [31] focused on two classical single-core caching algorithms, LEAST-RECENTLY-USED (LRU) [37] and FURTHEST-IN-FUTURE (FIF) [11], and showed these algorithms are unboundedly worse than the optimal algorithm OPT in the free-interleaving model<sup>3</sup>. In the free-interleaving

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<sup>&</sup>lt;sup>1</sup>This problem is also called "paging" in the literature [31]. We use "multicore caching" because it more accurately reflects the problem studied in this paper.

<sup>&</sup>lt;sup>2</sup>For a cost-minimization problem, an online algorithm has a competitive ratio of c if its cost on any input never exceeds c times the cost of an optimal offline algorithm for the same input (up to an additive constant).

 $<sup>{}^{3}</sup>$ LRU is an online caching algorithm that evicts the leastrecently-requested page. FIF is an offline caching algorithm that evicts the page that will be requested furthest in the future. Both algorithms evict pages only when the cache is full and there is a request to a page not in the cache. In the multicore setting, ties

model, FIF evicts the page furthest in the future in terms of the number of requests. In the single-core setting, LRU is k-competitive (where k is the size of the cache) [37], and FIF is the optimal algorithm [11]. Kamali and Xu [25] further confirmed the intuition that multicore caching is much harder than singlecore caching and showed that all lazy algorithms are equivalently non-competitive against OPT. An online caching algorithm is *lazy* [32] if it 1) evicts a page only if there is a miss 2) evicts no more pages than the misses at each timestep, 3) in any given timestep, does not evict a page that incurred a hit in that timestep, and 4) evicts a page only if there is no space left in the cache<sup>4</sup>. Lazy algorithms capture natural and practical properties of online algorithms. Common caching strategies such as LRU and First-In-First-Out (FIFO) are clearly lazy. Unfortunately, the competitive ratio of this huge class of algorithms is bounded and grows with the length of the input.

The existing negative results for competitive analysis consider a cost model in which the goal is to minimize the number of misses. Nevertheless, they extend to the case when the objective is the total number of timesteps to answer all requests [25]. We focus on the latter measure in this paper because it is more practical, as we will explain in detail in Section 2.

At a high level, the divergence between performance of algorithms for multicore and single-core caching stems from the power of the adversary to adapt to online algorithms and to generate inputs that are particularly tailored to harm the schedule of online algorithms. For these adversarial inputs, the implicit scheduling of lazy algorithms causes periods of "high demand" in which the cache of the algorithm is congested (cores request many different pages). Meanwhile, an optimal offline algorithm avoids these high-demand periods by delaying cores in an "artificial way". These adversarial inputs highlight the inherent pessimistic nature of competitive analysis.

**Beyond worst-case analysis.** The highlystructured nature of the worst-case inputs suggests that competitive analysis might not be suitable for studying multicore caching algorithms and motivates the study of alternatives to competitive ratio. There are two main reasons to go beyond competitive analysis for analysis of multicore caching algorithms. First, competitive analysis is overly pessimistic and measures performance on worst-case sequences that are unlikely to happen in practice. In contrast, measures of typical performance

are more holistic than worst-case analysis, which dismisses all other sequences. Second, competitive analvsis does not help to separate online algorithms for multicore caching because no practical algorithm can compete with an optimal offline algorithm [25]. Therefore, other measures are required to establish the advantage of one online algorithm over others. Many alternative measures have been proposed for single-core caching [43, 42, 29, 26, 16, 14, 45, 44, 12]. For a survey of measures of online algorithms, we refer the reader to [21, 28, 15]. In particular, bijective analysis [5, 7, 8] is a natural measure that directly compares online algorithms and has been used to capture the advantage of LRU over other online single-core caching algorithms on inputs with "locality of reference" [5, 7]. Despite these results, as we will show, bijective analysis has restrictions when it comes to multicore caching.

**1.1 Contributions** We take the first steps beyond competitive analysis for multicore caching by extending bijective analysis to a stronger measure named cyclic analysis and demonstrating how to apply cyclic analysis to analyze multicore caching algorithms. The pessimistic nature of competitive analysis demonstrates the need for alternative measures of online algorithms.

**Cyclic analysis.** We introduce cyclic analysis, a measure that captures the benefits of bijective analysis and offers additional flexibility which we will demonstrate in our analysis of multicore caching algorithms. Cyclic analysis generalizes bijective analysis by directly comparing two online algorithms over *all inputs*. Traditional bijective analysis compares algorithms by partitioning the universe of inputs based on input length and drawing bijections between inputs in the same partition [4, 7, 5, 20]. Cyclic analysis relaxes this requirement by allowing bijections between inputs of different lengths. This flexibility allows for alternative proof methods for showing relationships between algorithms.

We show that all lazy [31, 32] algorithms are equivalent under cyclic analysis. More interestingly, we show the strict advantage of any lazy algorithm over Flush-When-Full (FWF) under cyclic analysis (FWF evicts all pages upon a miss on a full cache). In the single-core setting, the advantage of lazy algorithms over FWF is strict and trivial: for any sequence, the cost of LRU is no more than FWF. In the multicore setting, however, such separation requires careful design and mapping with a bijection on the entire universe of inputs (Theorem 4.1) under cyclic analysis.

Separation of LRU via cyclic analysis. Our main contribution is to show the strict advantage of a variant of LRU over all other lazy algorithms under cyclic analysis combined with a measure of locality

can happen; both LRU and FIF break ties arbitrarily.

 $<sup>^{4}</sup>$ Lazy algorithms are often called "demand paging" in the systems literature [35]. Algorithms with properties 1-3 (but not necessarily 4) are called "honest" algorithms [31].

(Theorem 5.2). Although LRU is equivalent to all other lazy algorithms without restriction on the inputs under cyclic analysis, it performs strictly better in practice [36]. This is due to the locality of reference that is present in real-world inputs [3, 19, 18]. In order to capture the advantage of LRU, we apply cyclic analysis on a universe that is restricted to inputs with locality of reference [3] and show that LRU is strictly better than any other lazy algorithm.

**Map.** The remainder of the paper is organized as follows. Section 2 describes the model of multicore caching and provides definitions used in the rest of the paper. Section 3 introduces cyclic analysis and establishes some useful properties of this measure. Section 4 applies cyclic analysis to establish the advantage of lazy algorithms over non-lazy FWF. Section 5 shows the advantage of LRU over all other lazy algorithms under cyclic analysis on inputs with locality of reference. Section 6 reviews related models of multicore caching, and Section 7 includes a few concluding remarks. All omitted proofs can be found in the full version.

#### 2 Problem definition

This section reviews the free-interleaving model [31, 27] of multicore caching and the cost models that are used in this paper. The free-interleaving model is inspired by real-world architectures and captures the essential aspects of the multicore caching problem.

Assume we are given a multicore processor with p cores labeled  $P_1, P_2, \ldots, P_p$  and a shared cache with k pages  $(k \gg p)$ .

Input description. An input to the *multicore* caching problem is formed by p online sequences  $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_p)$ . Each core  $P_i$  must serve its corresponding request sequence  $\mathcal{R}_i = \langle \sigma_{i,1}, \ldots, \sigma_{i,n_i} \rangle$  made up of  $n_i$ page requests. The total number of page requests is therefore  $n = \sum_{1 \leq i \leq p} n_i$ . Given a page (or sequence of pages)  $\alpha$  and a number of repetitions r, let  $\alpha^r$ denote r repetitions of requests to  $\alpha$ . We assume that for all values of i, the length of the request sequence  $n_i$  is arbitrarily larger than k. That is, we assume that  $k \in \Theta(1)$ , which is consistent with the common assumption that parameters like k and  $\tau$  are constant compared to the length of the input.

All requests  $\sigma_{i,j}$  are drawn from a finite universe of possible pages U. Throughout this paper, we assume that request sequences for different cores may share requests to the same page. In practice, cores may share their requests because of races, or concurrent accesses to the same page.

Serving inputs. Page requests arrive at discrete timesteps. The requests issued by each core should be

served in the same order that they appear and in an online manner. More precisely, for all  $i, j \geq 1$ , core  $P_i$ must serve request  $\sigma_{i,j}$  before  $\sigma_{i,j+1}$ , and  $\sigma_{i,j+1}$  is not revealed before  $\sigma_{i,j}$  is served. The multicore processor may serve at most p page requests in parallel (up to one request per core<sup>5</sup>). Each page request must be served as soon as it arrives. To serve a request to some page  $\sigma_{i,i}$  in sequence  $\mathcal{R}_i$ , core  $P_i$  either has a **hit**, when  $\sigma_{i,i}$ is already in the cache, or incurs a **miss** when  $\sigma_{i,j}$ is not present in the cache. In case of a miss, the requested page should be fetched into the cache. It takes  $\tau$  timesteps to fetch a page into the cache, where  $\tau$  is an integer parameter of the problem. During these timesteps,  $P_i$  cannot see any of its forthcoming requests, that is,  $\sigma_{i,j+1}$  is not revealed to  $P_i$  before  $\sigma_{i,j}$  is **fully fetched**. In case some other core  $P^* \neq P_i$  is already fetching the page when the miss occurs,  $P_i$  waits for less than  $\tau$  timesteps until the page is fully fetched to the cache.

Free-interleaving model. A multicore caching algorithm  $\mathcal{A}$  reads requests from request sequences in parallel and is defined by its eviction decisions at each timestep. If a core misses while the cache is full,  $\mathcal{A}$  must evict a page to make space for the requested page before fetching it. We continue the convention [24, 31] that when a page is evicted, the cache cell that previously held the evicted page is unused until the replacement page is fetched. Finally, the processor serves requests from different request sequences in the same timestep in some fixed order (e.g., by core index). In today's multicore systems, requests from multiple cores may reach a shared cache simultaneously. If one core is delayed due to a request to a page not in the cache, other cores may continue to make requests. Figure 1 contains an example of serving an input with LEAST-RECENTLY-USED (LRU) [37, 24, 31] under free interleaving.

**Schedule.** Multicore caching differs from singlecore caching because of the scheduling component as a result of the fetch delay. The fetch delay slows down cores at different rates depending on the misses they experience, and requests with the same index on different cores may be served at different times depending on previous evictions. In other words, the eviction strategy implicitly defines a *schedule*, or an ordering in which the requested pages are served by an algorithm. Given an input  $\mathcal{R}$  defined by p sequences, the schedule of a caching algorithm can be represented with a copy of  $\mathcal{R}$  in which some requests are repeated.

<sup>&</sup>lt;sup>5</sup>In practice, a single instruction of a core may involve more than one page, but we assume that each request is to one page in order to model RISC architectures with separate data and instruction caches [31].

These extra requests captures the timestep at which the processor serves requests from that input sequence using the caching algorithm. That is, a schedule has all the same requests as the corresponding input, but repeats page requests upon a miss until the page has been fully fetched.

The schedule produced by LRU in the example input in Figure 1 is the underlined request at each timestep (a formal definition of a schedule can be found in Section 5).

Cost model. We use the *total time* to measure algorithm performance and denote the cost that an algorithm  $\mathcal{A}$  incurs on input  $\mathcal{R}$  with  $\mathcal{A}(\mathcal{R})$ . The noncompetitiveness results from prior work in terms of the number of misses also hold under the total time [31, 25].

DEFINITION 2.1. (TOTAL TIME) The total time an algorithm  $\mathcal{A}$  takes to serve an input  $\mathcal{R}$  is the sum of the timesteps it takes for all cores to serve their respective request sequences. That is, the total time  $\mathcal{A}(\mathcal{R}) =$  $\sum_{1 \leq i \leq p} \mathcal{A}(\mathcal{R}_i) \text{ where } \mathcal{A}(\mathcal{R}_i) \text{ denotes the timesteps } P_i \text{ took}$ to serve  $\mathcal{R}_i$  with algorithm  $\mathcal{A}$ .

Total time combines aspects from both makespan and the number of misses, the two cost measures in previous studies of multicore caching [31, 27]. The makespan is the maximum time it takes any core to complete its request sequence, and hence is bounded above by total time. Specifically, the total time is monotonically increasing with respect to both the number of misses and the makespan.

The total time is a more realistic measure of performance than the number of misses because it determines performance in terms of the time that it takes to serve the input. In contrast, the number of misses does not directly correspond with the time to serve an input because a miss may take less than  $\tau$  steps to fetch the page if it is already in the process of being fetched by another core. The total time also captures aspects of algorithm performance that are not addressed by makespan. In particular, makespan does not capture the overall performance of all cores. For example, a solution in which all cores complete at timestep t has a better makespan than a solution in which one core completes at timestep t + 1 while the rest complete much earlier, e.g. at timestep t/2. The second solution is preferred in practice (and also under the total time) as most cores are freed up earlier.

#### Cyclic analysis for online problems 3

We define a new analysis measure called *cyclic analy*sis inspired by bijective analysis [6, 4, 7, 5, 20] and ex- DEFINITION 3.1. (CYCLIC ANALYSIS) We say that an

plore alternative paths to showing relationships between algorithms under cyclic analysis via a relaxed measure called "natural surjective" analysis. Cyclic analysis extends the advantages of bijective analysis to online problems with multiple input sequences.

**Overview.** Although traditional bijective analysis has been applied to compare single-core caching algorithms, it requires modification to capture the notion of "input length" in multicore caching. Since each request sequence in an input for multicore caching may have a different length in terms of the number of requests, there are multiple ways to define the length of an input. It is not clear which definition of length is most natural or correct for multicore caching.

Furthermore, partitioning the input space based on the number of requests in an input as in bijective analysis for single-core paging may be overly restrictive for multicore caching, because the time it takes to serve inputs of the same length (in terms of the number of requests) may differ depending on the algorithm. In multicore caching, the time depends on the interleaving of the multiple request sequences. Cyclic analysis addresses these issues by removing the restriction that bijections should be drawn between inputs of the same length.

At a high level, in order to show a relationship between two algorithms  $\mathcal{A}$  and  $\mathcal{B}$  under bijective analysis or cyclic analysis, one must define a mapping between inputs and their costs under different algorithms. One way to model mappings between inputs with different costs is with a *input-cost graph*. Given algorithms  $\mathcal{A}$  and  $\mathcal{B}$ , an input-cost graph is an infinite directed graph where the nodes represent inputs and there exists an edge from input  $\mathcal{R}_1$  to input  $\mathcal{R}_2$  if and only if  $\mathcal{A}(\mathcal{R}_1) \leq \mathcal{B}(\mathcal{R}_2)$ . In order to show the advantage of algorithm  $\mathcal{A}$  over  $\mathcal{B}$ , traditional bijective analysis partitions the (infinite) graph of inputs into finite subgraphs, each formed by inputs of the same length. Within each partition, the bijection relating  $\mathcal{A}$  to  $\mathcal{B}$  defines a set of cycles such that each vertex is in exactly one cycle of finite length (cycles may have length one, i.e. they may be self-loops). Cyclic analysis relaxes the requirement that all subgraphs in the partition must be finite, but also requires that each node in each induced subgraph must have an in-degree and out-degree of one. That is, each node in the induced subgraph is part of a cycle.

Measure definition and discussion. Let  $\mathcal{I}$ denote the (infinite) set of all inputs, and for an algorithm  $\mathcal{A}$  and input  $\mathcal{R} \in \mathcal{I}$ , let  $\mathcal{A}(\mathcal{R})$  denote the cost  $\mathcal{A}$  incurs while serving  $\mathcal{R}$ . The notation in our discussions of cyclic analysis is inspired by [4].

Timestep $(t)$	Cache before $t$	$\mathcal{R}_1,\mathcal{R}_2$	Status	Schedule $S_{\mathcal{R},\mathrm{LRU}}[t]$
0		$a_1 a_2 a_1 a_5$	$P_1$ misses, starts fetching $a_1$	$(a_1, a_3)$
		$a_3 a_4 a_5 a_2$	$P_2$ misses, starts fetching $a_3$	
1	$\perp \perp \perp \perp$	$\underline{a_1}a_2a_1a_5$	$P_1$ is fetching $a_1$	$(a_1, a_3)$
		$a_3a_4a_5a_2$	$P_2$ is fetching $a_3$	
2	$\perp \perp \perp \perp$	$\underline{a_1}a_2a_1a_5$	$P_1$ completes fetching $a_1$	$(a_1, a_3)$
		$a_3 a_4 a_5 a_2$	$P_2$ completes fetching $a_3$	
3	$a_1a_3 \bot \bot$	$a_1 \underline{a_2} a_1 a_5$	$P_1$ misses, starts fetching $a_2$	$(a_2, a_4)$
		$a_3 \underline{a_4} a_5 a_2$	$P_2$ misses, starts fetching $a_4$	
4	$a_1a_3 \bot \bot$	$a_1 \underline{a_2} a_1 a_5$	$P_1$ is fetching $a_2$	$(a_2, a_4)$
		$a_3 a_4 a_5 a_2$	$P_2$ is fetching $a_4$	
5	$a_1a_3 \bot \bot$	$a_1 \underline{a_2} a_1 a_5$	$P_1$ completes fetching $a_2$	$(a_2, a_4)$
		$a_3 \underline{a_4} a_5 a_2$	$P_2$ completes fetching $a_4$	
6	$a_1 a_3 a_2 a_4$	$a_1a_2\underline{a_1}a_5$	$P_1$ has a hit for $a_1$	$(a_1, a_5)$
		$a_3a_4\underline{a_5}a_2$	$P_2$ misses, starts fetching $a_5$	
			$(a_3 \text{ is the least-recently-used page and evicted})$	
7	$a_1 \perp a_2 a_4$	$a_1a_2a_1\underline{a_5}$	$P_1$ misses, waits for $a_5$	$(a_5,a_5)$
		$a_3a_4\underline{a_5}a_2$	$P_2$ is fetching $a_5$	
8	$a_1 \perp a_2 a_4$	$a_1 a_2 a_1 \underline{a_5}$	$P_1$ completes serving $a_5$	$(a_5,a_5)$
		$a_3a_4\underline{a_5}a_2$	$P_2$ completes fetching (and serving) $a_5$	
9	$a_1 a_5 a_2 a_4$	$a_1 a_2 a_1 a_5$	$P_1$ has completed $\mathcal{R}_1$	$(\perp, a_2)$
		$a_3a_4a_5\mathbf{a_2}$	$P_2$ has a hit for $a_2$ , completes $\mathcal{R}_2$	

Figure 1: Example of execution of LRU on the input  $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$ , with  $\mathcal{R}_1 = \langle a_1 a_2 a_1 a_5 \rangle$  and  $\mathcal{R}_2 = \langle a_3 a_4 a_5 a_2 \rangle$ . The cache size is k = 4 and the fetch delay is  $\tau = 3$ . We use  $\perp$  in the cache to denote an empty slot or slot reserved for a page currently being fetched.

If a request incurs a miss, we repeat it in the schedule at most  $\tau$  times (or however long it takes to be fetched, if some other processor already requested it but it has not yet been brought to cache). For example, in timestep 7, we wait two timesteps for  $a_5$  to come to the cache for  $P_1$  because there were two more steps until  $a_5$  was brought to the cache by  $P_2$ .

In the "Cache before t" column, we keep track of the state of the cache before each timestep. The rightmost column is the schedule generated by LRU serving  $\mathcal{R}$ . The makespan of  $\mathcal{R}$  under LRU is 10.

The schedule for the two cores  $P_1$  and  $P_2$  is defined respectively with  $\langle a_1, a_1, a_1, a_2, a_2, a_2, a_1, a_5, a_5, \bot \rangle$  and  $\langle a_3, a_3, a_3, a_4, a_4, a_4, a_5, a_5, a_5, a_5, a_2 \rangle$ .

online algorithm  $\mathcal{A}$  is **no worse** than online algorithm  $\mathcal{B}$  under **cyclic analysis** if there exists a bijection  $\pi: \mathcal{I} \leftrightarrow \mathcal{I}$  satisfying  $\mathcal{A}(\mathcal{R}) \leq \mathcal{B}(\pi(\mathcal{R}))$  for each  $\mathcal{R} \in \mathcal{I}$ . We denote this by  $\mathcal{A} \preceq_c \mathcal{B}$ . Otherwise we denote the situation by  $\mathcal{A} \not\leq_c \mathcal{B}$ . Similarly, we say that  $\mathcal{A}$  and  $\mathcal{B}$  are the same according to cyclic analysis if  $\mathcal{A} \preceq_c \mathcal{B}$  and  $\mathcal{B} \preceq_c \mathcal{A}$ . This is denoted by  $\mathcal{A} \equiv_c \mathcal{B}$ . Finally we say  $\mathcal{A}$  is better than  $\mathcal{B}$  according to cyclic analysis if  $\mathcal{A} \preceq_c \mathcal{B}$  and  $\mathcal{B} \not\leq_c \mathcal{A}$ . We denote this by  $\mathcal{A} \prec_c \mathcal{B}$ .

Bijective analysis is defined similarly, except that the input universe is partitioned based on the length of inputs, and bijections need to be drawn between inputs inside each partition. In contrast, cyclic analysis allows mapping arbitrary sequences to each other. Bijective analysis and cyclic analysis have several benefits over competitive analysis [7]. Specifically, they:

• capture overall performance. If  $\mathcal{A} \leq_c \mathcal{B}$ , every "bad" input for algorithm  $\mathcal{A}$  corresponds to another input

for algorithm  $\mathcal{B}$  which is at least as bad. Hence, the performance of algorithms is evaluated over all request sequences rather than a single worst-case sequence.

- avoid comparing to an offline algorithm. Competitive analysis is inherently pessimistic as it compares online algorithms based on their worst-case performance against a powerful adversary. This pessimism is especially pronounced in multicore caching where an offline algorithm can "artificially" miss on some pages in order to schedule sequences in a way to minimize its total cost. This scheduling power is a great advantage for OPT as shown in [31]. Instead, we use cyclic analysis because compares online algorithms directly without involving an offline algorithm.
- can incorporate assumptions about the universe of inputs. Cyclic analysis can also define relationships between algorithm performance on a subset  $S \subset \mathcal{I}$

of inputs. For example, applying cyclic analysis to a restricted universe of inputs with locality of reference has been used to separate LRU from other algorithms in the single-core setting [4, 7]. Since LRU exploits locality of reference, analyzing inputs with locality may yield a better understanding of the performance of algorithms. Most other measures such as competitive ratio are unable to separate LRU from other lazy algorithms [4].

As mentioned above, bijective analysis, as defined for single-core caching [4, 7], requires partitioning the universe of inputs  $\mathcal{I}$  into finite sets of inputs of the same length. For multicore caching, however, this partitioning is not necessary nor well-defined. In fact, for many online problems, the length of input is not necessarily a measure of "difficulty", as trivial request (e.g., repeating requests to a page) can artificially increase the length. As such, there is no priory reason to draw bijections between sequences of the same length.

For problems such as single-core caching and list update [7], where the input is formed by a single sequence, the length of the input is simply the length of the sequence. In multicore caching, however, the length of inputs is not well-defined as multiple sequences are involved. Should the length be the sum of the number of requests or a vector of lengths for each request To address these issues, cyclic analysis sequence? generalizes the finite partitions of bijective analysis to the entire universe of inputs. This would give cyclic analysis a flexibility that makes it possible to study other problems under this measure. We note that, the restrictive nature of bijective analysis not only makes it hard to study algorithms under this measure, but also can cause situations that many algorithms are not comparable at all. The following example illustrates the restriction of bijective analysis when compared to cyclic analysis:

**Example.** Consider two algorithms  $\mathcal{A}$  and  $\mathcal{B}$  for an online problem P (with a single sequence as its input). Assume the costs of  $\mathcal{A}$  and  $\mathcal{B}$  are the same over all inputs, except for four sequences. Among these four, suppose that two sequences  $\sigma_1$  and  $\sigma_2$  have the same length m and we have  $\mathcal{A}(\sigma_1) = 10$  and  $\mathcal{A}(\sigma_2) = 40$  while  $\mathcal{B}(\sigma_1) = 20$  and  $\mathcal{B}(\sigma_2) = 30$ . For inputs of length m, there is no way to define a bijection that shows advantage of one algorithm over another. So, the two algorithms are incomparable under bijective analysis. Next, assume for sequences  $\sigma_3$  and  $\sigma_4$  we have  $\mathcal{A}(\sigma_3) = 20, \mathcal{A}(\sigma_4) = 30, \mathcal{B}(\sigma_3) = 40$ , and  $\mathcal{B}(\sigma_4) = 20$ . The following mappings shows  $\mathcal{A} \prec_c \mathcal{B}: \sigma_1 \to \sigma_1, \sigma_2 \to \sigma_3, \sigma_3 \to \sigma_4$ , and  $\sigma_4 \to \sigma_2$ .

Bounding inputs with the same cost. In

order for cyclic analysis to be a meaningful measure, there must not be an infinite number of inputs that achieve the same cost. To be more precise, for the universe of inputs  $\mathcal{I}$  and an algorithm  $\mathcal{A}$ , let  $\mathcal{A}(\mathcal{I})$  be the corresponding multiset of costs associated with inputs in  $\mathcal{I}$ .

DEFINITION 3.2. (BOUNDED-SHARED-COST PROPERTY) A cost measure for an online problem satisfies the bounded-shared-cost property if and only if for any algorithm  $\mathcal{A}$  and for all unique costs  $m \in \mathcal{A}(\mathcal{I})$ , the set of inputs that achieve that cost is bounded.

If a cost measure does not satisfy the boundedshared-cost property, it is possible to prove contradicting results under cyclic analysis. That is, if there are infinitely many inputs that achieve each cost, for any algorithms  $\mathcal{A}, \mathcal{B}$ , it is possible to define bijections such that  $\mathcal{A} \prec_c \mathcal{B}$  and  $\mathcal{B} \prec_c \mathcal{A}$ .

In the case of multicore caching, the total time and makespan cost models both have the bounded-sharedcost property while the miss count and the closely related miss rate do not. For example, the infinitely many sequences that only request some page  $\alpha$  (e.g.  $\alpha$ ,  $\alpha\alpha\alpha$ ,  $\alpha\alpha\alpha$ , ...) all have cost one under miss count, but all have different costs under total time and makespan.

The following lemma guarantees that cyclic analysis has the "to-be-expected" property that if algorithm  $\mathcal{A}$ is better than  $\mathcal{B}$ ,  $\mathcal{B}$  is not better than  $\mathcal{A}$ . In the case of bijective analysis, this property easily follows from the fact that bijections are drawn in finite sets (formed by inputs of the same length). Since the bijections in cyclic analysis are defined in an infinite space, a more careful analysis is required.

LEMMA 3.1. Given algorithms  $\mathcal{A}, \mathcal{B}$  for a problem satisfying the bounded-shared-cost property, it is not possible that  $\mathcal{A} \prec_c \mathcal{B}$  and  $\mathcal{B} \prec_c \mathcal{A}$  at the same time.

Proof. If  $\mathcal{A} \prec_c \mathcal{B}$ , by Definition 3.1, there must exist an input  $\sigma \in \mathcal{I}$  such that  $\mathcal{A}(\sigma) < \mathcal{B}(\pi(\sigma))$ . Let  $\sigma$  be the input with the smallest cost that differs between  $\mathcal{A}, \mathcal{B}$ , and let  $\mathcal{I}_{\mathcal{A}}^{\mathcal{A}(\sigma)}, \mathcal{I}_{\mathcal{B}}^{\mathcal{A}(\sigma)} \subset \mathcal{I}$  be the sequences that have cost at most  $\mathcal{A}(\sigma)$  in  $\mathcal{A}(\mathcal{I}), \mathcal{B}(\mathcal{I})$ , respectively. By the bounded-shared-cost property,  $|\mathcal{I}_{\mathcal{A}}^{\mathcal{A}(\sigma)}|$  and  $|\mathcal{I}_{\mathcal{B}}^{\mathcal{A}(\sigma)}|$  are both bounded and  $|\mathcal{I}_{\mathcal{A}}^{\mathcal{A}(\sigma)}| > |\mathcal{I}_{\mathcal{B}}^{\mathcal{A}(\sigma)}|$ . It is impossible to define another function  $\phi$  such that  $\mathcal{B} \preceq_c \mathcal{A}$  because there are not enough inputs in  $\mathcal{I}_{\mathcal{B}}^{\mathcal{A}(\sigma)}$  to map to all inputs in  $\mathcal{I}_{\mathcal{A}}^{\mathcal{A}(\sigma)}$  such that the cost of each input under  $\mathcal{B}$  is at most the cost of the corresponding input under  $\mathcal{A}$ .

Similarly, if  $\mathcal{A} \prec_c \mathcal{B}$ , then  $\mathcal{A} \not\equiv_c \mathcal{B}$  for problems with the bounded-shared-cost property. Additionally, cyclic analysis has the transitive property: if  $\mathcal{A} \preceq_c \mathcal{B}$  and  $\mathcal{B} \leq_c \mathcal{C}$ , then  $\mathcal{A} \leq_c \mathcal{C}$ . The bounded-sharedcost property guarantees that each node in the inputcost graph has infinite out-degree but finite in-degree because each input has infinitely many inputs that cost more than it and finitely many inputs that cost less than it.

Relation of surjectivity to cyclic analysis. In the remainder of the section we will discuss the role of surjective mappings as an intermediate step before defining a bijective mapping between infinite sets. In traditional bijective analysis, since the input set is finite because of the length restriction, any surjective mapping must also be bijective. In some problems, including multicore caching, it may be easier to define a surjective mapping between the inputs. We will first show that a class of surjective mappings can be converted into bijective mappings.

Suppose we have a surjective but not necessarily injective mapping between two infinite sets  $f: X \to Y$ . For all positive integers  $m \in \mathbb{N}$ , let  $X_m \subseteq X, Y_m \subseteq Y$ be subsets of the pre-image and image respectively such that exactly m elements in  $X_m$  map to one element in  $Y_m$ . That is, given some  $m, x \in X_m$  implies that there are m - 1 other elements  $x_1, x_2, \ldots, x_{m-1} \neq x$  such that for  $i = 1, \ldots, m - 1$ ,  $f(x) = f(x_i)$ . Each  $X_m, Y_m$ is an element of a partition of the pre-image and image, respectively.

#### DEFINITION 3.3. (NATURAL SURJECTIVE MAPPING) Given a surjective function $f: X \to Y$ , f is **natural** if and only if for all $m \in \mathbb{N}$ , the partitions $X_m$ and $Y_m$

are either empty or infinite.

For example, the function  $f : \mathbb{N} \to \mathbb{N}, f(x) = \lfloor x/2 \rfloor$ is a natural surjective mapping (assuming  $0 \in \mathbb{N}$ ) because exactly two elements in the pre-image map to each element in the image. In contrast,  $g : \mathbb{Z} \to \mathbb{N}, g(x) = |x|$  is not natural because there is only one element in  $X_1$  and  $Y_1$  at x = y = 0.

We introduce *natural surjective* (NS) analysis, a technique to compare algorithms under cyclic analysis using an intermediate surjective *but not injective* mapping. The formalization is almost identical to Definition 3.1, but the function  $\pi$  needs only to be a natural surjective function. We use  $\leq_s$  to denote the relation between two algorithms under NS analysis. In the rest of the paper, we will refer to natural surjective functions and natural surjective analysis as surjective functions and surjective analysis, respectively.

LEMMA 3.2. ("UNZIPPING" EQUIVALENCE) Let algorithms  $\mathcal{A}, \mathcal{B}$  be algorithms for a problem with the bounded-shared-cost property. If  $\mathcal{A} \leq_s \mathcal{B}$  under a natural surjective mapping, then  $\mathcal{A} \prec_c \mathcal{B}$ .



Figure 2: Example of unzipping  $X_2, Y_2$  in a natural surjective mapping.

*Proof.* At a high level, we will describe how to convert a natural surjective function f into a bijective mapping  $f_b$  by "unzipping" any many-to-one mappings in each partition. At a high level, the new mapping  $f_b$  "remaps" elements in the preimage to elements in the image.

Let  $X_m, Y_m$  be the pre-image and image of a nonempty mapping-based partition for any fixed  $m \in \mathbb{N}$ . Suppose we order the elements in  $Y_m$  from lowest to highest and let  $y_i$  be the *i*-th largest element in  $Y_m$ . The elements in any set  $Y_m$  can be ordered because of the bounded-shared-cost property. Given an element  $y_i^m \in Y_m$ , let the corresponding elements in the preimage be  $x_{i,j} \in X_m$  for  $j = 1, 2, \ldots, m$  in some order. Since  $\mathcal{A}(x_{i,j}) \leq \mathcal{B}(y_i^m)$  for all i, j (by the definition of surjective analysis), for any  $i, j, \mathcal{A}(x_{i,j}) \leq \mathcal{B}(y_z^m)$  for z > i. Therefore, we define a new bijective mapping  $f_b$  based on f such that  $f_b(x_{i,j}) = y_{mi+j-1}^m$ . The new mapping  $f_b$  satisfies the property that for all  $\sigma \in \mathcal{I}$ ,  $\mathcal{A}(\sigma) \leq \mathcal{B}(f_b(\sigma))$ .

As shown in the example in Figure 2, we can convert a natural surjective mapping to a bijective one by "unzipping" the mapping and maintaining the relative order of inputs.

The relationship between surjective analysis and cyclic analysis allows for different paths to proving relationships between algorithms. In traditional bijective analysis, we had to define a direct bijection between two algorithms because all surjections are bijections in finite sets of the same size. Natural surjective analysis is a potentially easier proof technique that is equivalent to cyclic analysis.

#### 4 Cyclic analysis for multicore caching

It is straightforward to show that all lazy multicore caching algorithms are equivalent under cyclic analysis (see Proposition A.1 for a full proof). Therefore, to show a separation between two algorithms, we analyze a variant of FWF that flushes (empties) the entire cache if it incurs a miss when the cache is full. In what follows, we show the advantage of lazy algorithms over FWF. While this result is not surprising, the techniques used in the proofs prepare the reader for the more complicated proof in the next section.

LEMMA 4.1. Assume p = 2. Consider two lazy caching algorithms  $\mathcal{A}$  and  $\mathcal{B}$  which have the same eviction policy starting at the same timestep t and cache contents at t except for one page x that is present in the cache of  $\mathcal{A}$ and absent in the cache of  $\mathcal{B}$ . If  $\mathcal{A}$  and  $\mathcal{B}$  incurred the same cost up until timestep t, we have  $\mathcal{A} \prec_{c} \mathcal{B}$ .

*Proof.* At a high level, we will define a surjective cyclic mapping on the input space with cycles of length 2. For inputs where x is never requested before being evicted,  $\mathcal{A}$  and  $\mathcal{B}$  perform similarly. We assume these inputs are mapped to themselves and ignore them (the cycles associated with these inputs are self-loops). In the remainder of the proof, we assume x is requested at timestep t before being evicted. At timestep t,  $\mathcal{A}$  has a hit on the request to x while  $\mathcal{B}$  incurs a miss. As a result, the schedule of the two algorithms (i.e., the order at which they serve the requests) becomes different after serving x and hence there is no guarantee that  $\mathcal{A}$  has less cost that  $\mathcal{B}$ .

We define a bijection b in a way that the schedule of  $\mathcal{A}$  for any input R is similar to that of  $\mathcal{B}$  for serving  $b(\mathcal{R})$ . The bijection that we define creates cycles of length 2: if  $\mathcal{R}' = b(\mathcal{R})$  then  $\mathcal{R} = b(\mathcal{R}')$ ; we denote this by  $\mathcal{R} \leftrightarrow \mathcal{R}'$ .

Let  $P_1$  and  $P_2$  denote the two cores and let  $\begin{cases} .. & \sigma_1 \\ .. & \sigma_2 \end{cases}$  denote the *continuation* of a sequence where  $P_1$  asks for sequence  $\sigma_1$  and  $P_2$  asks for  $\sigma_2$  from time t onward. We define the bijection based on two cases. In both cases, one of the cores, say  $P_2$ , has a request to page xat time t and hence  $\mathcal{A}$  and  $\mathcal{B}$  perform differently on the continuation of the sequence. Assume the contents of the caches of  $\mathcal{A}$  and  $\mathcal{B}$  at time t are respectively  $H \cup \{x\}$ and H.

**Case 1:**  $P_1$  requests a page  $q \notin H$ .

Recall that  $P_2$  asks for x at time t, so the input can be written as  $\mathcal{R} = \begin{cases} \cdots q\sigma \\ \cdots x\sigma' \end{cases}$  for some  $\sigma$  and  $\sigma'$ . We define  $\mathcal{R}' = \begin{cases} \cdots q\sigma \\ \cdots x\sigma' \end{cases}$ . To show the mapping  $\mathcal{R} \leftrightarrow \mathcal{R}'$  is a valid mapping we need to show  $\mathcal{A}(\mathcal{R}) \leq \mathcal{B}(\mathcal{R}')$  and  $\mathcal{A}(\mathcal{R}') \leq \mathcal{B}(\mathcal{R})$ . First, we show  $\mathcal{A}(\mathcal{R}) \leq \mathcal{B}(\mathcal{R}')$ . On input  $\mathcal{R}$ ,  $\mathcal{A}$  has a miss on q and a hit on x at time t; so,  $\mathcal{A}$  starts serving  $\sigma$  and  $\sigma'$  at timesteps  $t + \tau$  and t + 1, respectively, it serves  $\sigma$  exactly  $\tau - 1$  timesteps later than  $\sigma'$ . On input  $\mathcal{R}'$ ,  $\mathcal{B}$  has a miss on both x and q at time t. It incurs an additional  $\tau - 1$  hits on x after fetching it. So,  $\mathcal{B}$  starts serving  $\sigma$  and  $\sigma'$  at timesteps  $t + \tau + (\tau - 1)$ and  $t+\tau$ , respectively. In other words, it serves  $\sigma$  exactly  $\tau - 1$  timesteps later than  $\sigma'$ . The content of the cache of

 $\mathcal{A}$  and  $\mathcal{B}$  is the same for serving  $\sigma$  and  $\sigma'$ . We conclude that the number of misses (and hence total time) of  $\mathcal{B}$ in serving  $\sigma$  and  $\sigma'$  in R is the same as  $\mathcal{A}$  in  $\mathcal{R}'$ . For the first requests to q and x in  $\mathcal{R}$ ,  $\mathcal{A}$  incurs one miss (and total time  $\tau + 1$ ) while  $\mathcal{B}$  incurs two misses (and total time  $3\tau - 1$ ) for the first requests to  $x^{\tau}$  and q in  $\mathcal{R}'$ . We conclude that  $\mathcal{A}(\mathcal{R}) < \mathcal{B}(\mathcal{R}')$ . To complete the proof in Case 1, we should show  $\mathcal{A}(\mathcal{R}') \leq \mathcal{B}(\mathcal{R})$ . When  $\mathcal{A}$  serves  $\mathcal{R}'$ , it incurs  $\tau$  hits on  $x^{\tau}$  and one miss on q; as such, it starts serving  $\sigma$  and  $\sigma'$  at the same time  $t + \tau$ . On the other hand, when  $\mathcal{B}$  serves  $\mathcal{R}$ , it incurs a miss on both q and x and starts serving  $\sigma$  and  $\sigma'$  at the same time  $t + \tau$ . So, the two algorithms incur the same cost for serving  $\sigma$  and  $\sigma'$ . Moreover,  $\mathcal{A}$  one miss and  $\tau$  hits (and total time  $2\tau$ ) for serving  $x^{\tau}$  and q while  $\mathcal{B}$  incurs two misses (and total time  $2\tau$ ) for serving q and x, so  $\mathcal{A}(\mathcal{R}') = \mathcal{B}(\mathcal{R}).$ 

**Case 2:**  $P_1$  asks for a page  $a \in H$ .

So, the input can be written as  $\mathcal{R} = \begin{cases} ... & a\sigma \\ ... & x\sigma' \end{cases}$  for some sequence of requests  $\sigma$  and  $\sigma'$ . We define  $\mathcal{R}' =$ for  $\int .. x\sigma$  $\begin{cases} \vdots & \overset{\iota \cup}{a^\tau \sigma'} \\ \vdots & a^\tau \sigma' \end{cases} \text{ To show the mapping } \mathcal{R} \, \leftrightarrow \, \mathcal{R}' \text{ is a valid}$ mapping we first show  $\mathcal{A}(\mathcal{R}) \leq \mathcal{B}(\mathcal{R}')$ .  $\mathcal{A}$  starts serving both  $\sigma$  and  $\sigma$  in  $\mathcal{R}$  at t+1 because  $\mathcal{A}$  has hits on both a and x. On the other hand,  $\mathcal{B}$  has a miss on x and a hit on a when serving all copies of  $\tau$ . That means, it starts serving both  $\sigma$  and  $\sigma'$  in  $\mathcal{R}'$  at the same time  $t + \tau$ . The content of the cache of the two algorithms is also the same (x is now in the cache of  $\mathcal{B}$ ). So,  $\mathcal{A}$  and  $\mathcal{B}$  incur the same number of misses (and total time) for both  $\sigma$  and  $\sigma'$ . For the prefixes a and x in  $\mathcal{R}$ ,  $\mathcal{A}$  incurs 0 misses (and total time 2); for the prefixes  $a^{\tau}$  and x in  $\mathcal{R}', \mathcal{B}$  incurs 1 miss (and total time  $2\tau$ ). We conclude  $\mathcal{A}(\mathcal{R}) < \mathcal{B}(\mathcal{R}')$ . Next, we show  $\mathcal{A}(\mathcal{R}') \leq \mathcal{B}(\mathcal{R})$ .  $\mathcal{A}$  has hits on all requests in  $a^{\tau}$  and x in  $\mathcal{R}'$ , i.e., it serves  $\sigma$ and  $\sigma'$  at timesteps t+1 and  $t+\tau$ , respectively. That is, it serves  $\sigma$  exactly  $\tau - 1$  units later than  $\sigma'$ .  $\mathcal{B}$ , on the other hand, has a hit at a and a miss at x in  $\mathcal{R}$ , i.e. it serves  $\sigma$  and  $\sigma'$  at times t + 1 and  $t + \tau$ , respectively. So, the two algorithms incur the same cost for  $\sigma$  and  $\sigma'$ . For the prefixes  $a^{\tau}$  and x,  $\mathcal{A}$  incurs 0 misses and total time  $\tau + 1$ . For the prefixes a and x,  $\mathcal{B}$  incurs 1 miss and total time  $\tau + 1$ . We conclude that  $\mathcal{A}(\mathcal{R}') \leq \mathcal{B}(\mathcal{R})$ . 

We show the advantage any lazy algorithm  $\mathcal{A}$  over non-lazy FWF by comparing their cache contents at each timestep.

THEOREM 4.1. Any lazy algorithm  $\mathcal{A}$  is strictly better than FWF under cyclic analysis for p = 2, that is,  $\mathcal{A} \prec_c \mathrm{FWF}.$ 

*Proof.* Let  $FWF_i$  be a variant of FWF which, instead

of flushing the cache, evicts i pages from the cache; these i pages are selected according to  $\mathcal{A}$ 's eviction policy. That is, the algorithm evicts i pages that  $\mathcal{A}$  evicts when its cache is full (as an example, if  $\mathcal{A}$  is LRU, the algorithm evicts the i least-recently-used pages). We will show  $\mathcal{A} \prec_c$  FWF by transitivity of bijection. In particular, we show

$$\mathcal{A} = \mathrm{FWF}_1 \prec_c \ldots \prec_c \mathrm{FWF}_{k-1} \prec_c \mathrm{FWF}_k = \mathrm{FWF}.$$

Let  $\operatorname{FWF}_i^t$  be an algorithm that applies  $\operatorname{FWF}_i$  for the first t timesteps and  $\operatorname{FWF}_{i+1}$  for timesteps after and including t+1. If we can show  $\operatorname{FWF}_i^{t+1} \prec_c \operatorname{FWF}_i^t$ for all t, again by transitivity of bijection, we get  $\operatorname{FWF}_i \prec_c \operatorname{FWF}_{i+1}$ . We note that  $\operatorname{FWF}_i^{t+1}$  and  $\operatorname{FWF}_i^t$ differ in serving at most one request at time t, and they have the same eviction strategy for the remainder of the input. If the cores do not incur a miss at time t, both algorithms perform similarly. For sequences for which there is a miss at time t, there will be one less page in the cache of  $\operatorname{FWF}_i^t$  compared to  $\operatorname{FWF}_i^{t+1}$ . Therefore,  $\operatorname{FWF}_i^{t+1} \prec_c \operatorname{FWF}_i^t$  by Lemma 4.1.  $\Box$ 

As the bijection in the proofs illustrates, the main insight of cyclic analysis is the direct comparison of algorithms by drawing mappings between inputs of different lengths. In contrast to the single-core setting, inputs of the same length (in the number of requests) in multicore caching may take different amounts of time, so we define bijections based on the schedule (and therefore length in time) rather than the number of requests. In the next section, we use the same idea of mapping sequences with different lengths in requests but similar schedules.

#### 5 Advantage of LRU with locality of reference

To demonstrate how to use cyclic analysis to separate algorithms, this section sketches the separation of LRU from all other lazy algorithms on inputs with locality of reference via cyclic analysis. Along the way, we demonstrate how to use surjective analysis to establish relations between algorithms under cyclic analysis. In practice, LRU (and its variants) are empirically better than all other known caching algorithms [36] because sequences often have temporal locality.

The full proofs for this section can be found in the full version.

5.1 Preliminaries First, we will formalize the notion of a schedule from Section 2, which represents an algorithm's eviction decisions by repeating requests in an input on a miss. We will use the schedule to later define locality of reference. Throughout this section, let  $\mathcal{A}$  be a caching algorithm and  $\mathcal{R}$  be an input. DEFINITION 5.1. (SCHEDULE) The schedule  $S_{\mathcal{R},\mathcal{A}} = \{S_{\mathcal{R}_1,\mathcal{A}},\ldots,S_{\mathcal{R}_p,\mathcal{A}}\}$  is another input where each request sequence is defined as the implicit schedule that  $\mathcal{A}$ generated while serving  $\mathcal{R}$ . That is,  $S_{\mathcal{R}_i,\mathcal{A}}[t]$  is the request that core  $P_i$  serves at timestep t under  $\mathcal{A}$ . Also,  $S_{\mathcal{R},\mathcal{A}}$  is the same as  $\mathcal{R}$  with each miss repeated at most  $\tau-1$  times (as many repetitions as it takes to resolve the given miss, which might be less than  $\tau - 1$  if the page was already in the process of being fetched). We use  $S_{\mathcal{R}_i,\mathcal{A}}[t_1,t_2]$  (for all i) to denote all requests (including repetitions due to misses) made by  $P_i$  between timesteps  $t_1$  and  $t_2$  (inclusive).

We use the formal definition of schedule to discuss dividing up an input under  $\mathcal{A}$  based on its schedule up until some timestep.

#### DEFINITION 5.2. (SCHEDULE PREFIX AND SUFFIX)

Let  $n_{\mathcal{R},\mathcal{A}}$  be the time required for  $\mathcal{A}$  to serve  $\mathcal{R}$ . Given an integer timestep  $j < n_{\mathcal{R},\mathcal{A}}$ , we define parts of the schedule that will be served before, after, and during timestep j + 1.

Informally, the schedule prefix  $S_{j,\mathcal{R},\mathcal{A}}^{pre}$  is all the requests served up to timestep j with repetitions matching scheduling delay, the schedule at timestep j + 1,  $S_{\mathcal{R},\mathcal{A}}[j+1]$ , is all requests served at timestep j + 1, and the schedule suffix  $S_{j,\mathcal{R},\mathcal{A}}^{suf}$  is all requests served after timestep j + 1 with repetitions matching scheduling delay. Note that  $S_{\mathcal{R},\mathcal{A}}^{pre}$  or  $S_{\mathcal{R},\mathcal{A}}^{suf}$  may be empty. When the timestep j and/or algorithm  $\mathcal{A}$  are clear from context, we will drop them from the schedule notation.

#### DEFINITION 5.3. (REQUEST PREFIX AND SUFFIX)

Let  $\mathcal{R}^{\leq j,\mathcal{A}}$  be all subsequences from  $\mathcal{R}$  served up to timestep j,  $\mathcal{R}^{>j,\mathcal{A}}$  be all subsequences from  $\mathcal{R}$  served after timestep j, and  $r_{j+1}$  be the requests at timestep j+1. For simplicity, we define the **request prefix** as  $\mathcal{R}^{pre} = \mathcal{R}^{\leq j,\mathcal{A}}$  and **request suffix** as  $\mathcal{R}^{suf} = \mathcal{R}^{>j+1,\mathcal{A}}$ when  $j, \mathcal{A}$  are understood from context.

The request prefix and suffix formalizes the analysis technique from Section 4 of defining mappings based on the continuation of the input after some timestep.

Using the LRU example in Figure 1 when j = 4,  $\mathcal{R}_1^{\text{pre}} = a_1 a_2$ ,  $\mathcal{R}_2^{\text{pre}} = a_3 a_4$  because those are the pages that have been requested until timestep 4. Since at timestep 5 all cores are fetching requests,  $r_{j+1} = \emptyset$ . Also,  $\mathcal{R}_1^{\text{suf}} = a_1 a_5$  and  $\mathcal{R}_2^{\text{suf}} = a_5 a_2$  because those are the requests remaining after timestep 5. Similarly,  $\mathcal{S}_{\mathcal{R}_1}^{\text{pre}} = a_1 a_1 a_1 a_2$  and  $\mathcal{S}_{\mathcal{R}_2}^{\text{pre}} = a_3 a_3 a_3 a_4$ . At timestep 4, both cores are fetching, so  $r_{j+1} = (a_2, a_4)$ . The suffix is the schedule for timesteps after 4, so  $\mathcal{S}_{\mathcal{R}_1}^{\text{suf}} = a_2 a_1 a_5 a_5$ and  $\mathcal{S}_{\mathcal{R}_2}^{\text{suf}} = a_4 a_5 a_5 a_5 a_2$ . Locality of reference and the Max-Model. We will restrict the space of all inputs with the "Max-Model", an experimentally-validated model of locality of reference that limits the number of distinct pages in subsequences of an input with a concave function [3].

We define a window of size w in the multicore setting as p runs of consecutive requests of length w (one for each core). The Max-Model for multicore caching is the same as in single-core caching except that it considers windows over all cores.

In the Max-Model for multicore caching, an input  $\mathcal{R}$  is **consistent** with some increasing concave function f if the number of distinct pages in any window of size w is at most f(w), for any  $w \in \mathbb{N}$  [3]. That is, a function  $f : \mathbb{N} \to \mathbb{R}^+$  is **concave** if f(1) = p, and  $\forall n \in \mathbb{N} : f(n+1) - f(n) \leq f(n+2) - f(n+1)$ . In the Max-Model, we also require that f is surjective on the integers between p and its maximum value.

It is easy to adapt cyclic analysis to the Max-Model by restricting to inputs consistent with a concave function f (denoted by  $\mathcal{I}^{f}$ ). Let  $\mathcal{A} \preceq^{f}_{c} \mathcal{B}$  denote that  $\mathcal{A}$ is no worse than  $\mathcal{B}$  on  $\mathcal{I}^{f}$  under cyclic analysis. Similarly, let  $\mathcal{A} \preceq^{f}_{s} \mathcal{B}$  denote that  $\mathcal{A}$  is no worse than  $\mathcal{B}$  on  $\mathcal{I}^{f}$ under surjective analysis.

**5.2** Advantage of LRU on inputs with locality In the rest of the section, we will show that LRU is no worse than sequences with locality under cyclic analysis by establishing a surjective mapping (Definition 3.3) and converting it into a bijective mapping (Lemma 3.2). The main technical challenge in the proof of the separation of LRU is that sequences with the same number of requests may have different schedules and therefore may differ significantly in their cost, even if they only differ in one request. We use cyclic analysis to avoid the restriction of comparing inputs of the same length and instead define a function to relate inputs of the same cost.

Along the way, we demonstrate how to use surjective analysis as a proof technique for comparing algorithms via cyclic analysis on the entire space of inputs as described in Section 3. The construction of the surjective mapping is inspired by a similar argument in the single-core setting by Angelopoulos and Schweitzer [7] which establishes a bijective mapping within finite partitions, but requires a more complex mapping based on schedules.

We will show that for every algorithm  $\mathcal{A}$ , LRU  $\preceq_s^f$  $\mathcal{A}$ . An arbitrary algorithm  $\mathcal{A}$  may be very different from LRU. Therefore, instead of defining a direct bijection, we will use intermediate algorithms  $\mathcal{B}_1, \ldots, \mathcal{B}_\ell$  such that  $\mathcal{A} \equiv \mathcal{B}_1 \succeq_s^f \ldots \succeq_s^f \mathcal{B}_i \succeq_s^f \ldots \succeq_s^f \mathcal{B}_\ell \equiv$  LRU. The result follows from the transitivity of the " $\preceq_s^f$ " relation. Intuitively, we construct algorithms "closer" to LRU at each step in the series as we will explain in Lemma 5.2. We formalize the notion of an algorithm  $\mathcal{A}$ 's "closeness" to LRU in terms of the evictions that it makes. An algorithm  $\mathcal{A}$  is **LRU**-*like* at timestep t if after serving all requests up to time t - 1, it serves all requests at time t as LRU would.

**Defining a surjective mapping between inputs.** At a high level, the proof proceeds by defining a surjection between similar sequences with two pages swapped. We define a "complement" of a sequence as a new sequence with certain pages swapped, and show properties of complements of sequences with locality required for our main proof.

DEFINITION 5.4. (COMPLEMENT [7]) Let  $\beta$ ,  $\delta$  denote two distinct pages in U, the universe of pages. Let  $\mathcal{R}_i[j]$ denote the *j*-th request in the *i*th request sequence of an input  $\mathcal{R}$ . The **complement** of  $\mathcal{R}_i[j]$  with respect to  $\beta$ and  $\delta$ , denoted by  $\overline{\mathcal{R}_i[j]}^{(\beta,\delta)}$ , is the function that replaces  $\beta$  with  $\delta$ , and vice versa. Formally,  $\overline{\mathcal{R}_i[j]}^{(\beta,\delta)} = \delta$ , if  $\mathcal{R}_i[j] = \beta$ ;  $\overline{\mathcal{R}_i[j]}^{(\beta,\delta)} = \beta$ , if  $\mathcal{R}_i[j] = \delta$ ; and  $\overline{\mathcal{R}_i[j]}^{(\beta,\delta)} = \mathcal{R}_i(j)$ , otherwise.

We use  $\overline{\mathcal{R}_i[j]}$  when  $\beta, \delta$  are clear from context. We denote each request sequence  $\mathcal{R}_i = \sigma_1^i \dots \sigma_{n_i}^i$ , where  $\mathcal{R}_i$  has  $n_i$  requests. For any sequence for a single core  $\mathcal{R}_i, \overline{\mathcal{R}_i} = \overline{\mathcal{R}_i[1]}, \dots, \overline{\mathcal{R}_i[n_i]}$ . For any multicore sequence  $\mathcal{R}, \overline{\mathcal{R}} = \{\overline{\mathcal{R}_1}, \dots, \overline{\mathcal{R}_p}\}$ . For any sequence  $\mathcal{R}_i$ , let  $\mathcal{R}_i[j_1, j_2]$  denote the (contiguous) subsequence of requests  $\sigma_{i,j_1}, \dots, \sigma_{i,j_2}$ . Also, we use  $\mathcal{R}_\alpha \cdot \mathcal{R}_\gamma$  to denote the concatenation of two sequences  $\mathcal{R}_\alpha, \mathcal{R}_\gamma$ .

We now extend a lemma from [7] about sequences with locality that we will use in our main theorem later. The lemma says that if a sequence  $\ldots \delta \ldots \beta \ldots \delta \ldots \beta \ldots$  exhibits locality of reference, then  $\ldots \delta \ldots \beta \ldots \beta \ldots \delta$  does as well.

LEMMA 5.1. Let  $\mathcal{R}$  be a sequence of requests consistent with f,  $\mathcal{A}$  be a caching algorithm, and  $n_{\mathcal{R},\mathcal{A}}$  be the time that it takes  $\mathcal{A}$  to serve  $\mathcal{R}$ . Let  $j \leq n_{\mathcal{R},\mathcal{A}}$  be an (integer) timestep such that  $S_{\mathcal{R},\mathcal{A}}[1,j]$  contains a request to  $\beta$ , and in addition,  $\delta$  does not appear in  $S_{\mathcal{R},\mathcal{A}}^{pre} = S_{\mathcal{R},\mathcal{A}}[1,j]$ after the last request to  $\beta$  in  $S_{\mathcal{R},\mathcal{A}}^{pre}$ .

after the last request to  $\beta$  in  $S_{\mathcal{R},\mathcal{A}}^{pre}$ . Let  $\mathcal{R}' = \mathcal{R}^{pre} \overline{\mathcal{R}^{suf}}$  denote the sequence  $\mathcal{R}^{\leq j,\mathcal{A}} \overline{\mathcal{R}}^{>j,\mathcal{A}}$ , and suppose that  $\mathcal{R}'$  is not consistent with f. Then  $\mathcal{R}^{suf}$  contains a request to  $\beta$ ; furthermore, no request to  $\delta$  in  $S_{\mathcal{R},\mathcal{A}}^{suf}$  ( $S_{\mathcal{R},\mathcal{A}}^{suf} = S_{\mathcal{R},\mathcal{A}}[j+1,n_{\mathcal{R},\mathcal{A}}]$ ) occurs earlier than the first request to  $\beta$  in  $S_{\mathcal{R},\mathcal{A}}^{suf}$ .

The following lemma guarantees that for any algorithm  $\mathcal{A}$  which may make a non-LRU-like eviction at the (j + 1)-th timestep of some  $\mathcal{R} \in \mathcal{I}^f$  (but will make

LRU-like evictions for the rest of the timesteps after j + 1), we can define an algorithm  $\mathcal{B}$  that makes the same decisions as  $\mathcal{A}$  up until timestep j of any sequence in  $\mathcal{I}^f$ , makes an LRU-like decision on the (j + 1)-th timestep, and is no worse than  $\mathcal{A}$  under surjective analysis.

LEMMA 5.2. Let  $\mathcal{I}^f$  be all inputs consistent with f and let j be an integer. Suppose  $\mathcal{A}$  is an algorithm with the property that for every input  $\mathcal{R} \in \mathcal{I}^f$ ,  $\mathcal{A}$  is LRU-like on timestep t + 1, for all  $t \geq j + 1$ . Then there exists an algorithm  $\mathcal{B}$  with the following properties:

- For every input R ∈ I<sup>f</sup>, B makes the same decisions as A on the first j timesteps while serving R (i.e., A and B make the same eviction decisions for each miss in requests up to and including time t).
- 2. For every input  $\mathcal{R} \in \mathcal{I}^f$ ,  $\mathcal{B}$  is LRU-like on  $\mathcal{R}$  at timestep t.
- 3.  $\mathcal{B} \preceq^f_s \mathcal{A}$ .

PROOF SKETCH. The main insight in this proof is the comparison of inputs with different numbers of page requests but the same cost under two different algorithms. If an algorithm  $\mathcal{A}$  makes a non-LRU-like decision at some timestep, we construct a surjection that maps it to a sequence with the same schedule under another algorithm  $\mathcal{B}$ .

At a high level, we use a "sequence reordering" mapping inspired by Lemma 2 of [7]. Let  $\mathcal{B}$  be an algorithm that matches the evictions of  $\mathcal{A}$  until time t, when it makes LRU-like evictions. Suppose at time t that  $\mathcal{A}$  evicted a page  $\sigma_{\text{NLRU}}$  and  $\mathcal{B}$  evicted a page  $\sigma_{\text{LRU}}$ . We construct  $\mathcal{B}$  to evict the same pages as  $\mathcal{A}$  on the remainder of the sequence.

We construct a surjective mapping  $\pi$  such that for any request sequence  $\mathcal{R}$ ,  $\mathcal{B}(\mathcal{R}) \leq \mathcal{A}(\pi(\mathcal{R}))$ . There are two main cases based on the continuation of the input after time t. At a high level, if an input has locality of reference, then there are not many requests to different pages. Now, if possible, we swap  $\sigma_{\text{LRU}}, \sigma_{\text{NLRU}}$  in the continuation of the input after time t since these will result in the same cost in the continuation.

**Case 1:** Swapping  $\sigma_{\text{LRU}}$ ,  $\sigma_{\text{NLRU}}$  in the continuation maintains locality. In this case,  $\mathcal{A}(\mathcal{R}) = \mathcal{B}(\pi(\mathcal{R}))$ because the different decisions at time t did not affect the number of misses (and therefore the total time) while serving the the rest of the input. Swapping the pages where  $\mathcal{A}$ ,  $\mathcal{B}$  differ in the continuation of the mapped-to input results in the same behavior.

**Case 2:** Swapping  $\sigma_{\text{LRU}}$ ,  $\sigma_{\text{NLRU}}$  in the continuation does not maintain locality. There are a few cases when swapping the two pages would disrupt locality.

- If there was a miss on another page before the first request to  $\sigma_{\text{NLRU}}$  in the continuation after time t, both algorithms would incur the same cost since the difference in decision does not affect the number of hits and misses in the rest of the input. In this case, we set  $\pi(\mathcal{R}) = \mathcal{R}$ , and  $\mathcal{B}(\mathcal{R}) = \mathcal{A}(\pi(\mathcal{R}))$ .
- If there was not a miss before the first request to  $\sigma_{\text{NLRU}}$  after time t,  $\mathcal{B}$  hits on the first request to  $\sigma_{\text{LRU}}$  in the continuation, and we remove requests in  $\pi(\mathcal{R})$  so that the schedule of  $\mathcal{B}$  serving  $\mathcal{R}$  matches the schedule of  $\mathcal{A}$  serving  $\pi(\mathcal{R})$ . Since the schedules match,  $\mathcal{B}(\mathcal{R}) = \mathcal{A}(\pi(\mathcal{R}))$ .
- The above two cases cover the entire codomain, but not the domain. For the remaining inputs, we can map them arbitrarily to inputs of higher cost such that there are no more than two inputs in the domain mapped to any input in the codomain. By construction,  $\mathcal{B}(\mathcal{R}) < \mathcal{A}(\pi(\mathcal{R}))$ . We present an example of generating such a mapping from an input  $\mathcal{R}$  under algorithms  $\mathcal{A}$  and  $\mathcal{B}$  given page  $\sigma$  in Figure 3.

Given any algorithm  $\mathcal{A}$ , we repeatedly apply Lemma 5.2 to construct a new algorithm  $\mathcal{B}$  which is LRU-like after some timestep t and is no worse than  $\mathcal{A}$ .

Let  $n_{\mathcal{R},\mathcal{A}}$  be the time it takes to serve input  $\mathcal{R}$  with  $\mathcal{A}$ , and let  $B_t$  be the class of algorithms that make LRUlike decisions on timesteps  $n_{\mathcal{R}} - t$  of every input  $\mathcal{R} \in \mathcal{I}^f$ .

LEMMA 5.3. For every algorithm  $\mathcal{A}$  there exists an algorithm  $\mathcal{B}_t \in B_t$  such that  $\mathcal{B}_t \preceq_s \mathcal{A}$ , and for every input  $\mathcal{R} \in \mathcal{I}^f$ ,  $\mathcal{B}_t$  makes the same decisions as  $\mathcal{A}$  during the first  $n_{\mathcal{R},\mathcal{A}} - t$  timesteps while serving  $\mathcal{R}$ .

For every lazy algorithm  $\mathcal{A}$ , Lemma 5.3 guarantees the existence of an algorithm  $\mathcal{B}$  that makes LRU-like decisions on all timesteps for any input in  $\mathcal{I}^f$  and is no worse than  $\mathcal{A}$ . The only algorithm with this property is exactly LRU.

THEOREM 5.1. For any lazy caching algorithm  $\mathcal{A}$ , LRU  $\leq_s^f \mathcal{A}$ .

We have defined a surjection from LRU to any other algorithm through intermediate algorithms that are progressively "closer to LRU". Therefore, we have shown that LRU is the best lazy algorithm under cyclic analysis via surjective analysis and therefore under cyclic analysis by combining Theorem 5.1 and Lemma 3.2.

THEOREM 5.2. For any lazy algorithm  $\mathcal{A}$ , LRU  $\prec_{c}^{f} \mathcal{B}$ .

We take the first steps beyond worst-case analysis for multicore caching with the separation of LRU from



Figure 3: An example of the mapping of an input  $\mathcal{R}$  under algorithm  $\mathcal{B}$  to  $\pi(\mathcal{R})$  under with  $\tau = 4$ . On the left, an input  $\mathcal{R}$  where  $a_1 = 2, b_2 = 2, a_2 = 7$ . The green boxes indicate hits on a page  $\sigma$  in  $\mathcal{B}$ 's cache but not in algorithm  $\mathcal{A}$ 's cache. On the right, we show the corresponding  $\pi(\mathcal{R})$ . The red boxes denote misses on  $\sigma$ .

all other lazy algorithms on inputs with locality via cyclic analysis. The main insight in the proof is to compare inputs of different lengths (in terms of the number of page requests) but the same schedule with a surjective mapping and then to convert the mapping into a bijection. Although we used it the case of multicore caching, cyclic analysis is a general analysis technique that may be applied to other online problems.

#### 6 Related multicore caching models

We review alternative models for multicore caching in order to explain why we use the free-interleaving model. Specifically, we discuss a class of models for multicore caching called fixed interleaving and the Schedule-Explicit model introduced by Hassidim [24]. At a high level, these models assume the order in which the requests are served is decided by the adversary. In practice, however, the schedule of an algorithm is implicitly defined through the eviction strategies [31, 30], so the free-interleaving model studied in this paper is more practical.

Existing work focuses on minimizing either the makespan of caching strategies or on minimizing the number of misses. In the case of single-core caching, minimizing the makespan and number of misses are equivalent as makespan is simply  $\tau$  times the number of misses. For multicore caching, however, there is no such direct relationship between makespan and number of misses. In this paper, we introduce the total time, a cost measure with benefits over both makespan and number of misses while capturing aspects of each.

Feuerstein and Strejilevich de Loma [39, 23] introduced multi-threaded caching as the problem of determining an optimal schedule in terms of the optimal interleaved request sequence from a set of individual request sequences from multiple cores. More precisely, given p request sequences  $\mathcal{R}_1, \ldots, \mathcal{R}_p$ , they study miss and makespan minimization for a "flattened" interleaving of all  $\mathcal{R}_i$ 's. Our work focuses on algorithms for page replacement rather than ordering (scheduling) of the input sequences. As mentioned, in practice, the schedule of page requests is embedded in the page-replacement algorithm.

Several previous works [10, 17, 27] studied multicore caching in the *fixed-interleaving model* (named by Katti and Ramachandran [27]). This model assumes each core has full knowledge of its future request sequence where the offline algorithm has knowledge of the interleaving of requests. The interleaving of requests among cores is the same for all caching algorithms and potentially adversarial (for competitive analvsis). Katti and Ramachandran [27] gave lower bounds and a competitive algorithm for fixed interleaving with cores that have full knowledge of their individual request sequences. In practice, cores do not have any knowledge about future requests, and do not necessarily serve requests at the same rate. Instead, they serve requests at different rates depending on whether they need to fetch pages to the cache.

Hassidim [24] introduced a model for multicore caching before free interleaving which we call the *Schedule-Explicit model* that allows offline algorithms to define an explicit schedule (ordering of requests) for the online algorithm. Given an explicit schedule, the online algorithm serves an interleaved sequence in the same way that a single-core algorithm does. The cost of the algorithm, measured in terms of makespan, is then compared against the cost of an optimal offline algorithm (which potentially serves the input using another schedule).

Both Schedule-Explicit and free-interleaving models include a fetch delay upon a miss, but Schedule-Explicit gives offline algorithms more power by allowing them to arbitrarily delay the start of sequences at no cost in terms of the number of misses (Theorem 3.1 of [24]). While Schedule-Explicit provides useful insight about serving multiple request sequences simultaneously, it leads to overly pessimistic results when minimizing the number of misses as it gives offline algorithms an unfair advantage.

Finally, competitive analysis for distributed systems illustrates the difficulty of multiple independent processes. For example, system nondeterminism in distributed algorithms [9] addresses nondeterminism in the system as well as in the input. Furthermore, recent work [13] confirms the difficulty that online algorithms face in "scheduling" multiple inputs in the distributed setting.

### 7 Conclusions

We take the first steps beyond worst-case analysis of multicore caching in this paper. In Theorem 5.2, we separated LRU from other algorithms on sequences with locality of reference. More generally, we introduced cyclic analysis and demonstrated its flexibility in the direct comparison of online algorithms. We expect cyclic analysis to be useful in the study of other online problems, and leave such application as future work.

We conclude by explaining that we are optimistic about multicore caching. Multicore caching is an important problem in online algorithms and motivated by computer architectures with hierarchical memory. Practitioners have extensively studied cache-replacement policies for multiple cores. The need for theoretical understanding of multicore caching will only grow as multicore architectures become more prevalent.

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# A Equivalence of lazy algorithms (from Section 4)

PROPOSITION A.1. If  $\mathcal{A}$  and  $\mathcal{B}$  are two arbitrary lazy algorithms,  $\mathcal{A} \equiv_{c} \mathcal{B}$ .

*Proof.* The proof is an extension of the proof of Theorem 3.3 from [5]. Let  $(n_1^t, \ldots, n_p^t)$  be the indices of  $\mathcal{R}_1, \ldots, \mathcal{R}_p$  being served at time t by  $\mathcal{A}, \mathcal{B}$ . Let  $n^t = \sum_{i=1}^p n_i^t$  be the number of requests served up until time t.

We prove by induction on time that for every  $t \geq 1$  that there is a bijection  $b^t : \mathcal{I}(n_1^t, \dots, n_p^t) \leftrightarrow$  $\mathcal{I}(n_1^t,\ldots,n_p^t)$  such that  $\mathcal{A}(\mathcal{R}) = \mathcal{B}(b^t(\mathcal{R}))$  for each  $\mathcal{R} \in \mathcal{I}(n_1^t, \dots, n_p^t)$ . For  $t \leq k\tau/p$ ,  $\mathcal{A}(\mathcal{R}) = \mathcal{B}(b^t(\mathcal{R}))$ trivially because  $\mathcal{A}$  and  $\mathcal{B}$  can only bring in up to k pages, so  $\mathcal{A}$  and  $\mathcal{B}$  behave the same and incur the same cost. Assume that for all  $n^t \leq h$  where  $h \geq k/p$ , we can define a bijection  $b^t : \mathcal{R}(n_1^t, \ldots, n_p^t)$  showing  $\mathcal{A}$  and  $\mathcal{B}$ are equivalent, where  $n_i^t$  is the number of requests up to time t of core  $P_i$ . We now show how to extend this bijection for n = h + 1. We define a new bijection  $b^{h+1}: \mathcal{I}(n_1^{h+1}, \ldots, n_p^{h+1}) \leftrightarrow \mathcal{I}(n_1^{h+1}, \ldots, n_p^{h+1})$ , which maps the continuations of each request sequence  $\mathcal{R}_i$ to the continuations of  $b^h(\mathcal{R}_i)$  in the image. By assumption, up to time h we have defined a bijection  $b^h$  that matches sequences for  $\mathcal{A},\,\mathcal{B}$  in terms of cost and schedule. That is, the number of pages k' < k being fetched at time t after serving  $\mathcal{R}$  by  $\mathcal{A}$  is the same as the number of pages being fetched at time t after serving

Let |P| = N be the number of distinct pages that any algorithm can request.

 $b^h(\mathcal{R})$  by  $\mathcal{B}$ .

Since there are i) k - k' possible next-hit requests in both  $\mathcal{A}$  and  $\mathcal{B}$  at time h + 1 and ii) the same number of cores not currently fetching in  $\mathcal{A}, \mathcal{B}$  at time h+1, we can arbitrarily biject these to each other in each  $\mathcal{R}_i$ . We also do the same for the N-k next-miss requests outside the cache and the misses on the k' requests being fetched for each  $\mathcal{R}_i$ .  $\mathcal{A}$  and  $\mathcal{B}$  incur the same cost in each mapping and maintain the same schedule,  $\mathcal{A} \equiv_b \mathcal{B}$ .  $\Box$ 

#### B Proofs for Lemmas in Section 5

**B.1** Formalizing LRU in the multicore setting In order to compare algorithms with LRU, we compare the state of the cache and the timestamps assigned to pages in the cache throughout the execution of different algorithms. At each timestep, LRU assigns integer *tags* [7] to each page in its cache to represent when they were most-recently accessed.

In general, an algorithm  $\mathcal{A}$  is **tag-based** if it uses tags to keep track of when pages were last accessed. Given an algorithm  $\mathcal{A}$  that uses tags, we denote the tag of some page  $\sigma$  in the cache at time t with  $\operatorname{tag}_{\mathcal{A}}[\sigma, \mathcal{S}_t]$ , where  $\mathcal{S}_t$  is the schedule of the input up to time t.

Since will be comparing LRU with arbitrary algorithms via surjective analysis, we will formalize *tagbased* LRU [7] in a shared cache. Tag-based LRU in the multicore setting is a straightforward extension of its definition in the single-core setting.

DEFINITION B.1. (TAG-BASED LRU ([7])) Tag-

**based** LRU assigns a set T of (integer) tags to each page in its cache to represent when they were most-recently accessed. Formally, for every page  $\sigma$  in the cache, let  $tag_{LRU}[\sigma, S_{\mathcal{R},\mathcal{A}}[t]]$  be the tag assigned to  $\sigma$  right after LRU has served requests up to timestep t. Tag-based LRU processes each request  $\sigma$  at each timestep  $\ell > t$  as follows:

- 1. If  $\sigma$  is a hit, LRU updates the tag of  $tag_{LRU}[\sigma, S_{\mathcal{R}, \mathcal{A}}[\ell]] = \ell$ .
- 2. If  $\sigma$  is a miss and not currently being fetched by another core, LRU will evict the page with the smallest tag (if the cache is full) and fetch  $\sigma$  to the cache while updating its tag for the next  $\tau$  timesteps as it is fetched.
- 3. If  $\sigma$  is a miss and currently being fetched by another core, LRU will not evict a page (since the eviction due to  $\sigma$  already happened) and the core that requested  $\sigma$  will stall for x steps until  $\sigma$  is brought to the cache.

#### **B.2** Proofs of Lemmas

LEMMA 5.1. Let  $\mathcal{R}$  be a sequence of requests consistent with f,  $\mathcal{A}$  be a caching algorithm, and  $n_{\mathcal{R},\mathcal{A}}$  be the time that it takes  $\mathcal{A}$  to serve  $\mathcal{R}$ . Let  $j \leq n_{\mathcal{R},\mathcal{A}}$  be an (integer) timestep such that  $\mathcal{S}_{\mathcal{R},\mathcal{A}}[1,j]$  contains a request to  $\beta$ , and in addition,  $\delta$  does not appear in  $\mathcal{S}_{\mathcal{R},\mathcal{A}}^{pre} = \mathcal{S}_{\mathcal{R},\mathcal{A}}[1,j]$ after the last request to  $\beta$  in  $\mathcal{S}_{\mathcal{R},\mathcal{A}}^{pre}$ . Let  $\mathcal{R}' = \mathcal{R}^{pre} \overline{\mathcal{R}^{suf}}$  denote the sequence

Let  $\mathcal{R}' = \mathcal{R}^{pre}\mathcal{R}^{suf}$  denote the sequence  $\mathcal{R}^{\leq j,\mathcal{A}}\overline{\mathcal{R}^{>j,\mathcal{A}}}$ , and suppose that  $\mathcal{R}'$  is not consistent with f. Then  $\mathcal{R}^{suf}$  contains a request to  $\beta$ ; furthermore,

no request to  $\delta$  in  $\mathcal{S}_{\mathcal{R},\mathcal{A}}^{suf}$  ( $\mathcal{S}_{\mathcal{R},\mathcal{A}}^{suf} = \mathcal{S}_{\mathcal{R},\mathcal{A}}[j+1,n_{\mathcal{R},\mathcal{A}}]$ ) occurs earlier than the first request to  $\beta$  in  $\mathcal{S}_{\mathcal{R},\mathcal{A}}^{suf}$ .

Proof. Since  $\mathcal{R}' = \mathcal{R}^{\text{pre}} \overline{\mathcal{R}^{\text{suf}}}$  is not consistent with f, there must exist indices  $j_{1,1}, j_{1,2}, \ldots, j_{p,1}, j_{p,2}$  such that for all  $i = 1, \ldots, p, j_{i,1} < j_{i,2} \leq n_i$  such that the number of distinct requests over all  $\mathcal{R}_i[j_{i,1}, j_{i,2}]$  exceeds  $f(j_2 - j_1 + 1)$  distinct pages. For any subsequence rin  $\mathcal{R}^{\text{suf}}$ ,  $\overline{\mathcal{R}}$  has the same number of distinct pages as r. Therefore, at least one of  $j_{i,1}, j_{i,2}$  must be such that  $j_{i,1} \leq t_{i,j} \leq j_{i,2}$  (where  $t_{i,j}$  is the index of some  $\mathcal{R}_i$  at time j under  $\mathcal{A}$ .

Suffices then to argue that for at least one  $i = 1, \ldots, p, \mathcal{R}_i[t_{i,j}, j_{i,2}]$  contains a request to  $\beta$  but not to  $\delta$ . For simplicity, we will specify a subsequence of one  $\mathcal{R}_i$  to mean over all  $i = 1, \ldots, p$ .

It is easy to see that  $\mathcal{R}_i[t_{i,j}, j_{i,2}]$  cannot contain requests to both  $\beta$  and  $\delta$ , nor can it contain requests to none of these pages: if either of these cases occurred, then  $\mathcal{R}_i[j_{i,1}, j_{i,2}]$  and  $\mathcal{R}_i[j_{i,1}, t_{i,j}]\overline{\mathcal{R}_i[t_{i,j}, j_{i,2}]}$  would contain the same number of distinct pages, which contradicts that  $\mathcal{R}$  is consistent with f. Note that  $\mathcal{R}_i[j_{i,1}, t_{i,j}]$ contains a request to  $\beta$  but not to  $\delta$ .

Now  $\mathcal{R}^{\text{pre}} \overline{\mathcal{R}^{\text{suf}}}$  must contain a request that does not appear in  $\mathcal{R}_i[j_{i,1}, j_{i,2}]$  and  $\delta$  is the only option. Therefore,  $\mathcal{R}^{\text{suf}}$  contains  $\beta$  but not  $\delta$ .

We advise the reader to first focus on the structure of the proof of Lemma 5.2 by skipping the proofs of the propositions, and then revisiting the details afterwards in Appendix C.

LEMMA 5.2. Let  $\mathcal{I}^f$  be all inputs consistent with f and let j be an integer. Suppose  $\mathcal{A}$  is an algorithm with the property that for every input  $\mathcal{R} \in \mathcal{I}^f$ ,  $\mathcal{A}$  is LRU-like on timestep t + 1, for all  $t \geq j + 1$ . Then there exists an algorithm  $\mathcal{B}$  with the following properties:

- For every input R ∈ I<sup>f</sup>, B makes the same decisions as A on the first j timesteps while serving R (i.e., A and B make the same eviction decisions for each miss in requests up to and including time t).
- 2. For every input  $\mathcal{R} \in \mathcal{I}^f$ ,  $\mathcal{B}$  is LRU-like on  $\mathcal{R}$  at timestep t.
- 3.  $\mathcal{B} \preceq^f_s \mathcal{A}$ .

*Proof.* First, we construct  $\mathcal{B}$  using  $\mathcal{A}$  on an input  $\mathcal{R} \in \mathcal{I}^f$ . At a high level,  $\mathcal{B}$  matches  $\mathcal{A}$ 's eviction decisions up to time j, makes an LRU-like decision at time j + 1, and matches  $\mathcal{A}$  in the remainder of the input. First, we require  $\mathcal{B}$  to make the same decisions as  $\mathcal{A}$  on all requests in  $\mathcal{R}^{\text{pre}}$ . If  $\mathcal{A}$  makes LRU-like decisions on all

misses at time j + 1, then  $\mathcal{B}$  makes the same LRU-like decision as  $\mathcal{A}$ , as well as the same decisions on all  $\mathcal{R}^{\text{suf}}$  as  $\mathcal{A}$ .

If  $\mathcal{A}$  makes a non-LRU-like decision at time j + 1, however, there must exist a pair of pages  $\sigma_{\text{LRU}}, \sigma_{\text{NLRU}} \in$ P where  $\sigma_{\text{LRU}} \neq \sigma_{\text{NLRU}}$  such that at timestep j + 1,  $\mathcal{A}$ evicts  $\sigma_{\text{NLRU}}$  from its cache, whereas  $\sigma_{\text{LRU}}$  is the leastrecently-used page in  $\mathcal{R}^{\text{pre}}$  (for now we assume that  $\mathcal{A}$ ,  $\mathcal{B}$  differ by only one page. The mapping in this lemma can be repeated for multiple pages, however.) If there are multiple non-LRU-like decisions at time j + 1, we can apply the same sequence-mapping technique for all of them.

We require that  $\mathcal{B}$  evicts  $\sigma_{\text{LRU}}$  in the remainder of the input if there is a miss. The tag of all other pages besides  $\sigma_{\text{NLRU}}$  is defined by the last time there were accessed, and the tag of  $\sigma_{\text{NLRU}}$  is the last time  $\sigma_{\text{LRU}}$ was accessed. More formally,  $\text{tag}_{\mathcal{B}}[\sigma_{\text{NLRU}}, \mathcal{S}_{\mathcal{R},\mathcal{A}}^{\text{pre}} \cdot s^{\mathcal{A}}] \leftarrow$  $\text{last}[\sigma_{\text{LRU}}, \mathcal{S}_{\mathcal{R},\mathcal{A}}^{\text{pre}}]$ , and  $\text{tag}_{\mathcal{B}}[\sigma, \mathcal{S}_{\mathcal{R},\mathcal{A}}^{\text{pre}} \cdot s^{\mathcal{A}}] \leftarrow \text{last}[\sigma, \mathcal{S}_{\mathcal{R},\mathcal{A}}^{\text{pre}}]$ for all pages  $\sigma \neq \sigma_{\text{NLRU}}$  in  $\mathcal{B}$ 's cache after time j + 1. We use  $\text{last}[\sigma_{\text{LRU}}, \mathcal{S}_{\mathcal{R},\mathcal{A}}^{\text{pre}}]$  to denote the *time* of the last access to  $\sigma_{\text{LRU}}$  in  $\mathcal{S}_{\mathcal{R},\mathcal{A}}^{\text{pre}}$ . After time j + 1, we require that  $\mathcal{B}$  is tag-based. Note that  $\mathcal{B}$  is completely online because it does not know the future.

The two algorithms differ in only one eviction:  $\mathcal{B}$  evicts  $\sigma_{\text{LRU}}$  instead of  $\sigma_{\text{NLRU}}$  (makes an LRU-like decision) and demotes the timestamp of  $\sigma_{\text{NLRU}}$  so that  $\sigma_{\text{NLRU}}$  is the least-recently-used page as  $\mathcal{B}$  prepares to serve the suffix  $\mathcal{R}^{\text{suf}}$ .

By construction,  $\mathcal{B}$  satisfies properties (1) and (2) of the lemma. In the rest of the proof, we will show property (3). Let  $\mathcal{S}_{\mathcal{I}^f,\mathcal{A}}$  be the set of schedules resulting from serving inputs with locality  $\mathcal{I}^f$  with  $\mathcal{A}$ .

We now define a mapping between inputs served by algorithms that differ on one eviction such that the two inputs have the same schedule.

DEFINITION B.2. (INVERSE INPUT ON ONE PAGE) Let  $\sigma$  be a page that algorithm  $\mathcal{B}$  hits on and  $\mathcal{A}$  misses on (for the first time after time j + 1) at time t > j + 1. Also, suppose that  $\mathcal{R}$  is an input with at least  $\tau$  repetitions of  $\sigma$  starting at time t under  $\mathcal{B}$ . We define the **inverse of**  $\mathcal{R}$  **in**  $\mathcal{B}$  **under**  $\mathcal{A}$  w.r.t.  $\sigma$ ,  $\mathcal{V}_{\sigma,\mathcal{R},\mathcal{A},\mathcal{B}}$ , as as follows:  $\mathcal{V}_{\sigma,\mathcal{R},\mathcal{A},\mathcal{B}}$  under  $\mathcal{A}$  generates the same schedule as  $\mathcal{R}$  under  $\mathcal{B}$ . Informally,  $\mathcal{V}_{\sigma,\mathcal{R},\mathcal{A},\mathcal{B}}$  removes all repetitions due to misses the first time  $\sigma$  is fetched after time j + 1.

Let  $\mathcal{R}$  be an input such that at least one core  $P_i$ requests  $\sigma$  at least  $\tau$  times starting at timestep t when served by  $\mathcal{B}$ . Formally, let  $P_i$  request  $\sigma \tau + a_i$  times starting at time t under  $\mathcal{B}$ , at index  $x_i$  through  $x_i + \tau + a_i$ in  $\mathcal{R}_i$ . In  $\mathcal{V}_{\sigma,\mathcal{R},\mathcal{A},\mathcal{B}}$ , we map those requests to a "shorter" input of repetitions: starting at index  $x_i$  in  $\mathcal{R}_i$ ,  $\mathcal{V}_{\sigma,\mathcal{R},\mathcal{A},\mathcal{B}}$  only has  $a_i + 1$  requests to  $\sigma$ . Furthermore, suppose any other core  $P_j \neq P_i$  repeats requests to  $\sigma$  at least  $b_j + a_j$ times starting at some timestep  $t + \tau - b_j$  (for  $0 < b_j \leq \tau$ ) and that they begin at index  $x_j$ . We map those requests to  $a_i + 1$  repetitions of  $\tau$  in  $\mathcal{V}_{\sigma,\mathcal{R},\mathcal{A},\mathcal{B}}$ . Note that for all  $i = 1, \ldots, p, a_i \geq 0$ .

The inverse  $\mathcal{V}_{\sigma,\mathcal{R},\mathcal{A},\mathcal{B}}$  is only defined for inputs that have at least  $\tau$  repetitions of  $\sigma$  at time t under  $\mathcal{B}$ . Let  $\mathcal{V}_{\sigma,\mathcal{I}^f,\mathcal{A},\mathcal{B}}$  be the set of inputs with locality where the inverse is defined for  $\mathcal{A}$ .

We present an example of generating  $\mathcal{V}_{\sigma,\mathcal{R},\mathcal{A},\mathcal{B}}$  from  $\mathcal{R}$  under  $\mathcal{A}$  and  $\mathcal{B}$  given page  $\sigma$  in Figure 3. In the example, we "shorten" the repetitions in  $\mathcal{V}_{\sigma,\mathcal{R},\mathcal{A},\mathcal{B}}$  such that  $\mathcal{A}$  serving  $\mathcal{V}_{\sigma,\mathcal{R},\mathcal{A},\mathcal{B}}$  generates the same schedule as  $\mathcal{B}$  serving  $\mathcal{R}$ . In  $\mathcal{V}_{\sigma,\mathcal{R},\mathcal{A},\mathcal{B}}$ ,  $p_1$  requests  $\sigma$  3 times  $(a_1 + 1)$  and  $p_2$  requests  $\sigma$  8 times  $(a_2 + 1)$ .

PROPOSITION B.3. Let f be an increasing concave function and  $\mathcal{A}$  be any caching algorithm. If an input  $\mathcal{R}$  is consistent with f, an input  $\mathcal{R}'$  based on  $\mathcal{R}$  that repeats any of its requests  $\sigma$  (immediately after  $\sigma$ ) is also consistent with f.

The only difference between  $\mathcal{R}'$  and  $\mathcal{R}$  is that  $\mathcal{R}'$ may have some repeated requests. Repeating requests does not increase the number of distinct pages in each window, so  $\mathcal{R}'$  must also be consistent with f.

Note that even if an input  $\mathcal{R}$  has locality of reference and has at least  $\tau$  repetitions of  $\sigma$  at time t under  $\mathcal{B}$ ,  $\mathcal{V}_{\sigma,\mathcal{R},\mathcal{A},\mathcal{B}}$  may not have locality of reference as it removes duplicates. Every local input that misses on  $\sigma$  at time t under  $\mathcal{A}$  has a corresponding input with repetitions to replicate  $\mathcal{A}$ 's schedule under  $\mathcal{B}$ , however, because creating the same schedule in  $\mathcal{B}$  requires only adding repetitions, which maintain locality (Proposition B.3).

We use surjective analysis via case analysis of the space of request inputs with locality as follows:

		$\mathcal{R}^{\mathrm{pre}}r_{j+1}\overline{\mathcal{R}^{\mathrm{suf}}}$	if $\mathcal{R}^{\mathrm{pre}}r_{j+1}\overline{\mathcal{R}^{\mathrm{suf}}}$ is consistent with $f$
			and $\mathcal{A}$ does not make an LRU-like
(B.1)			decision on $\mathcal{R}^{\operatorname{pre}}r_{j+1}$ .
		R	$\mathcal{R}^{\operatorname{pre}}r_{j+1}\overline{\mathcal{R}^{\operatorname{suf}}}$ is not consistent with $f$ ,
			and $\mathcal B$ incurs a miss before the first
(B.2)			request to $\sigma_{LRU}$ in $\mathcal{R}^{suf}$ .
	$\pi(\mathcal{P}) = \delta$	$\mathcal{V}_{\sigma,\mathcal{R},\mathcal{A},\mathcal{B}}$	$\mathcal{R}^{\text{pre}}r_{j+1}\overline{\mathcal{R}^{\text{suf}}}$ is not consistent with $f$ ,
	$\pi(\mathcal{K}) =$		${\mathcal B}$ does not incur a miss before the
			first request to $\sigma_{LRU}$ in $\mathcal{R}^{suf}$ ,
(B.3)			and $\mathcal{R} \in \mathcal{V}_{\sigma_{\mathrm{NLRU}}, \mathcal{I}^{f}, \mathcal{A}, \mathcal{B}}$
		$\mathcal{R}'$	$\mathcal{R}^{\operatorname{pre}}r_{j+1}\overline{\mathcal{R}^{\operatorname{suf}}}$ is not consistent with $f$ ,
			${\cal B}$ does not incur a miss before the
			first request to $\sigma_{LRU}$ in $\mathcal{R}^{suf}$ ,
(B.4)		l	and $\mathcal{R} \notin \mathcal{V}_{\sigma_{\mathrm{NLRU}},\mathcal{I}^{f},\mathcal{A},\mathcal{B}}$

where  $\overline{\mathcal{R}^{\text{suf}}}$  denotes the complement of  $\mathcal{R}^{\text{suf}}$  with respect to  $\sigma_{\text{LRU}}$  and  $\sigma_{\text{NLRU}}$  ( $\overline{\mathcal{R}^{\text{suf}}}^{(\sigma_{\text{LRU}},\sigma_{\text{NLRU}})}$ ). Additionally,  $\mathcal{R}'$  is another input such that  $\mathcal{A}$  serving  $\mathcal{R}'$  has a greater total time than  $\mathcal{B}$  serving  $\mathcal{R}$  (i.e.  $\mathcal{B}(\mathcal{R}) < \mathcal{A}(\mathcal{R}')$ ).

First, we show that  $\pi(\mathcal{R})$  accounts for all  $\mathcal{R} \in \mathcal{I}^f$ .

PROPOSITION B.4. The function  $\pi(\mathcal{R}) : \mathcal{I}^f \leftrightarrow \mathcal{I}^f$  is surjective and non-injective.

PROOF SKETCH. Cases 1-3 of  $\pi(\mathcal{R})$  account for the entire codomain but not the entire domain, because case (3) is surjective on that partition of the codomain. Therefore,  $\pi(\mathcal{R})$  is a natural surjective mapping because there are infinitely many inputs in Case B.4, so there are infinitely many one-to-one mappings in Cases 1-3, and then infinitely many two-to-one mappings from Case 4.

Now we will show that for every  $\mathcal{R} \in \mathcal{I}^f$ ,  $\mathcal{B}(\mathcal{R}) \leq \mathcal{A}(\pi(\mathcal{R}))$ . Again, we only consider the case where  $\mathcal{A}$  does not make an LRU-like request at time j + 1. We proceed by case analysis in Propositions B.6 and B.7. Since we will be comparing the cache contents of  $\mathcal{A}$  and  $\mathcal{B}$  by induction, we define the cache state of  $\mathcal{B}$  and  $\mathcal{A}$  as they serve  $\mathcal{R}$  and  $\pi(\mathcal{R})$ , respectively.

DEFINITION B.5. (CACHE STATE (INFORMAL, [7])) The **cache state** of an algorithm  $\mathcal{A}$  at any timestep t consists of the set of pages in the cache as well as the tag assigned to each page. For a more formal definition, see Definition C.1.

We choose tags at time j + 1 to make  $\mathcal{A}$  LRU-like and tag-based on the suffix of  $\mathcal{R}$  so that we can compare  $\mathcal{A}$  to  $\mathcal{B}$ .

PROPOSITION B.6. (CASE 1 OF  $\pi(\mathcal{R})$ ) If  $\mathcal{R}^{pre}r_{j+1}\overline{\mathcal{R}^{suf}}$  is consistent with  $f, \mathcal{B}(\mathcal{R}) = \mathcal{A}(\pi(\mathcal{R})).$ 

PROOF SKETCH. We prove the proposition by induction on the timestep  $\ell$ . We will show that the cache states of  $\mathcal{A}$  and  $\mathcal{B}$  are such that  $\mathcal{B}$  incurs a miss at time  $\ell$  on  $\mathcal{R}$  if and only if  $\mathcal{A}$  incurs a miss at time  $\ell$  on  $\pi(\mathcal{R})$ . We proceed by case analysis.

- **Case 1.** If none of the requests at time  $\ell$  are  $\sigma_{\rm NLRU}, \sigma_{\rm LRU}$ , then  $\mathcal{A}, \mathcal{B}$  have the same behavior and incur the same cost at time  $\ell$ . Therefore, the proposition holds for  $\ell + 1$ .
- **Case 2.** If  $\mathcal{B}$  sees a request to  $\sigma_{\text{LRU}}$  at time  $\ell$ , then  $\mathcal{A}$  sees a request to  $\sigma_{\text{NLRU}}$ . By the induction hypothesis, they have the same behavior with their respective  $\sigma_{\text{NLRU}}$ ,  $\sigma_{\text{LRU}}$ , and update their cache states to assign the same tag to their respective pages.
- **Case 3.** If  $\mathcal{B}$  sees a request to  $\sigma_{\text{NLRU}}$  at time  $\ell$ , then  $\mathcal{A}$  sees a request to  $\sigma_{\text{LRU}}$ , and we use a symmetric argument to Case 2.

PROPOSITION B.7. (CASES 2, 3, 4 OF  $\pi(\mathcal{R})$ ) If  $\mathcal{R}^{pre}r_{j+1}\overline{\mathcal{R}^{suf}}$  is not consistent with f, then  $\mathcal{B}(\mathcal{R}) \leq \mathcal{A}(\pi(\mathcal{R}))$ .

PROOF SKETCH. We proceed by case analysis on  $\pi(\mathcal{R})$ . By construction,  $\mathcal{A}$  and  $\mathcal{B}$  incur the same cost up until time j + 1. Their cache states differ only in that  $\mathcal{A}$ 's cache contains  $\sigma_{\text{LRU}}$  and  $\mathcal{B}$ 's cache contains  $\sigma_{\text{NLRU}}$ . Since  $\mathcal{R}^{\text{pre}}r_{j+1}\overline{\mathcal{R}}^{\text{suf}}$  is not consistent with f, Lemma 5.1 states that both  $\sigma_{\text{LRU}}$  and  $\sigma_{\text{NLRU}}$  must appear in the suffix  $\mathcal{R}^{\text{suf}}$  and that  $\sigma_{\text{NLRU}}$  must be requested earlier (in time) than  $\sigma_{\text{LRU}}$  in  $\mathcal{R}^{\text{suf}}$ .

- **Case 2 of**  $\pi(\mathcal{R})$ . If  $\mathcal{B}$  incurs a miss before the first request to  $\sigma_{\text{LRU}}$  in  $\mathcal{R}^{\text{suf}}$ ,  $\pi(\mathcal{R}) = \mathcal{R}$ . Both  $\mathcal{A}$  and  $\mathcal{B}$  incur a miss at time  $\ell$ , and replace  $\sigma_{\text{NLRU}}$  and  $\sigma_{\text{LRU}}$ , respectively. Therefore,  $\mathcal{A}$  and  $\mathcal{B}$  also have all the same eviction decisions after time  $\ell$  because they have matching cache states, so  $\mathcal{B}(\mathcal{R}) = \mathcal{A}(\pi(\mathcal{R}))$ .
- **Case 3 of**  $\pi(\mathcal{R})$ . Suppose that the first request to  $\sigma_{\text{LRU}}$  in  $\mathcal{R}^{\text{suf}}$  occurs at time t. If  $\pi(\mathcal{R}) = \mathcal{V}_{\sigma_{\text{NLRU}},\mathcal{R},\mathcal{A},\mathcal{B}}$ ,  $\mathcal{A}$  and  $\mathcal{B}$  do not incur any misses between times j + 1 and t. At time t,  $\mathcal{B}$  incurs a hit and  $\mathcal{A}$  incurs a miss. By definition of inverse,  $\mathcal{A}$  and  $\mathcal{B}$  so  $\mathcal{B}(\mathcal{R}) = \mathcal{A}(\pi(\mathcal{R}))$  because they have the same total time (repeated requests in  $\mathcal{B}$  to match the miss in  $\mathcal{A}$ ).

**Case 4 of**  $\pi(\mathcal{R})$ . If  $\pi(\mathcal{R}) = \mathcal{R}', \ \mathcal{B}(\mathcal{R}) < \mathcal{A}(\pi(\mathcal{R}))$  by construction of  $\mathcal{R}'$ .

We have shown in Propositions B.6 and B.7 that there exists a surjection  $\pi$  such that for all  $\mathcal{R} \in \mathcal{I}^f$ ,  $\mathcal{B}(\mathcal{R}) \leq \mathcal{A}(\pi(\mathcal{R}))$ .

LEMMA 5.3. For every algorithm  $\mathcal{A}$  there exists an algorithm  $\mathcal{B}_t \in \mathcal{B}_t$  such that  $\mathcal{B}_t \preceq_s \mathcal{A}$ , and for every input  $\mathcal{R} \in \mathcal{I}^f$ ,  $\mathcal{B}_t$  makes the same decisions as  $\mathcal{A}$  during the first  $n_{\mathcal{R},\mathcal{A}} - t$  timesteps while serving  $\mathcal{R}$ .

*Proof.* We proceed by induction on t. The lemma is trivially true for t = 0. Let  $\mathcal{B}_t \in B_t$  be an algorithm such that  $\mathcal{B}_t \leq_s \mathcal{A}$ , and for any input  $\mathcal{R} \in \mathcal{I}^f$ ,  $\mathcal{B}_t$  makes the same decisions as  $\mathcal{A}$  for the first  $n_{\mathcal{R},\mathcal{A}} - t$  timesteps while serving  $\mathcal{R}$ .

We show that the claim holds for t + 1 as well. From Lemma 5.2, there exists an algorithm  $\mathcal{B}$  such that  $\mathcal{B} \leq_s \mathcal{B}_t$ , and for every  $\mathcal{R} \in \mathcal{I}^f$ ,  $\mathcal{B}$  makes an LRU-like decision at time  $n_{\mathcal{R},\mathcal{A}} - t$ , and matches  $\mathcal{B}_t$  on the first  $n_{\mathcal{R},\mathcal{A}} - t - 1$  requests in  $\mathcal{R}$ .

Note that  $\mathcal{B}$  does not necessarily make LRU-like decisions for requests after  $n_{\mathcal{R},\mathcal{A}}-t+1$ . By the induction

hypothesis, there exists an algorithm  $\mathcal{B}'_t \in B_t$  such that i)  $\mathcal{B}'_t \preceq_s \mathcal{B}$ , and ii) for every  $\mathcal{R} \in \mathcal{I}^f$ ,  $\mathcal{B}'_t$  makes the same decisions as  $\mathcal{B}$  on the first  $n_{\mathcal{R},\mathcal{A}} - t$  timesteps of  $\mathcal{R}$ , and LRU-like decisions on the remaining timesteps. By definition,  $\mathcal{B}'_t \in B_{t+1}$ . We can reapply the induction hypothesis:  $\mathcal{B}'_t$  makes the same decisions as  $\mathcal{A}$  in the first  $n_{\mathcal{R},\mathcal{A}} - t - 1$  timesteps of  $\mathcal{R}$ , and so the lemma holds for t + 1.

#### C Proofs for Propositions in Section 5

PROPOSITION B.4. The function  $\pi(\mathcal{R}) : \mathcal{I}^f \leftrightarrow \mathcal{I}^f$  is surjective and non-injective.

*Proof.* Lemma 5.2 is trivially true if  $\mathcal{A}$  made only LRUlike requests at time j + 1 because  $\mathcal{A}$  and  $\mathcal{B}$  would be the same. Therefore, we will consider the case where  $\mathcal{A}$ makes a non-LRU-like eviction at time j + 1.

We proceed by cases following the definition of  $\pi(\mathcal{R})$ .

- **Case B.1.**  $\mathcal{A}$  also does not make an LRU-like eviction at time j + 1 on both  $\mathcal{R}$  and  $\pi(\mathcal{R})$ . Since the complement of  $\overline{\mathcal{R}^{\text{suf}}}$  is just  $\mathcal{R}^{\text{suf}}$ ,  $\pi(\pi(\mathcal{R})) = \mathcal{R}$ .
- **Case B.2.**  $\mathcal{R}^{\text{pre}}r_{j+1}\overline{\mathcal{R}^{\text{suf}}}$  is not consistent with f and  $\mathcal{B}$  incurs a miss before the first request to  $\sigma_{\text{LRU}}$  in  $\mathcal{R}^{\text{suf}}$ . Trivially,  $\pi(\pi(\mathcal{R})) = \mathcal{R}$  because  $\pi(\mathcal{R}) = \mathcal{R}$ .
- **Case B.3.** If  $\mathcal{R}^{\text{pre}}r_{j+1}\overline{\mathcal{R}^{\text{suf}}}$  is not consistent with f,  $\mathcal{B}$  does not incur a miss before the first request to  $\sigma_{\text{LRU}}$  in  $\mathcal{R}^{\text{suf}}$ , and  $\mathcal{R} \in \mathcal{V}_{\sigma_{\text{NLRU}},\mathcal{I}^{f},\mathcal{A},\mathcal{B}}$ , then  $\pi(\mathcal{R}) = \mathcal{V}_{\sigma_{\text{NLRU}},\mathcal{R},\mathcal{A},\mathcal{B}}$ . The set of all inverses from  $\mathcal{R} \in \mathcal{V}_{\sigma_{\text{NLRU}},\mathcal{I}^{f},\mathcal{A},\mathcal{B}}$  is all sequences in  $\mathcal{I}^{f}$  where  $\mathcal{R}^{\text{pre}}r_{j+1}\overline{\mathcal{R}^{\text{suf}}}$  is not consistent with f. From Definition B.2,  $\mathcal{V}_{\sigma_{\text{NLRU}},\mathcal{I}^{f},\mathcal{A},\mathcal{B}}$  is the set of all sequences with at least one request to  $\sigma$  at time t.
- **Case B.4.** If  $\mathcal{R}^{\text{pre}}r_{j+1}\overline{\mathcal{R}^{\text{suf}}}$  is not consistent with f,  $\mathcal{B}$  does not incur a miss before the first request to  $\sigma_{\text{LRU}}$  in  $\mathcal{R}^{\text{suf}}$ , and  $\mathcal{R} \notin \mathcal{V}_{\sigma_{\text{NLRU}},\mathcal{I}^{f},\mathcal{A},\mathcal{B}}$ , then  $\pi(\mathcal{R}) = \mathcal{R}'$ . Cases 1, 2, and 3 actually map to all of  $\mathcal{I}^{f}$ , but we require Case 4 because we have not yet accounted for all of the domain. Since we have already defined a mapping to all of the codomain in the first three cases, all we need is a corresponding input  $\mathcal{R}'$  such that  $\mathcal{B}(\mathcal{R}) \leq \mathcal{A}(\mathcal{R}')$ .

Therefore,  $\pi(\mathcal{R})$  is a natural surjective mapping because there are infinitely many inputs in Case B.4, so there are infinitely many one-to-one mappings in Cases 1-3, and then infinitely many two-to-one mappings from Case 4.  $\Box$ 

DEFINITION C.1. (CACHE STATE (FORMAL) [7]) Let  $C[\mathcal{A}, \mathcal{R}]$  be the **cache state** of algorithm  $\mathcal{A}$  after it has served input  $\mathcal{R}$ . The cache state consists of the set  $P[\mathcal{A}, \mathcal{R}]$  of pages in the cache after serving  $\mathcal{R}$ , as well as assigned tags  $tag_{\mathcal{A}}[\sigma, \mathcal{R}]$  equal to  $last_{\mathcal{A}}[\sigma, \mathcal{R}]$  for all  $\sigma \in P[\mathcal{A}, \mathcal{R}]$ .

For example  $C[\mathcal{A}, \mathcal{R}^{pre}r_{j+1}]$  is the cache state of  $\mathcal{A}$  after it has served requests up to time j + 1.

The **complement** of cache state  $C[\mathcal{A}, \mathcal{R}]$  with respect to  $\beta$  and  $\delta$ , denoted by  $\overline{C}[\mathcal{A}, \mathcal{R}]$  is a cache state in which:

- the set of pages is the set  $\overline{P[\mathcal{A}, \mathcal{R}]}$  (where  $\alpha$  is replaced with  $\beta$  and vice versa).
- tags are as in  $C[\mathcal{A}, \mathcal{R}]$  except for: if  $\beta \in \overline{P[\mathcal{A}, \mathcal{R}]}$ (resp. if  $\delta \in \overline{P[\mathcal{A}, \mathcal{R}]}$ ), then  $\beta$ 's tag in  $\overline{C}[\mathcal{A}, \mathcal{R}]$ is the tag of  $\delta$  in  $C[\mathcal{A}, \mathcal{R}]$  (resp. the tag of  $\beta$  in  $C[\mathcal{A}, \mathcal{R}]$ ).

PROPOSITION B.6. (CASE 1 OF  $\pi(\mathcal{R})$ ) If  $\mathcal{R}^{pre}r_{j+1}\overline{\mathcal{R}^{suf}}$  is consistent with  $f, \mathcal{B}(\mathcal{R}) = \mathcal{A}(\pi(\mathcal{R})).$ 

*Proof.* Let  $\mathcal{R}^{\leq \ell, \mathcal{A}}$  be the requests served by  $\mathcal{A}$  up to and including time  $\ell$ . Let  $ms(\mathcal{A}, \mathcal{R})$  be the makespan of  $\mathcal{A}$  on  $\mathcal{R}$ . We will show that for all  $j + 1 \leq \ell \leq ms(\mathcal{A}, \mathcal{R})$ , algorithm  $\mathcal{B}$  satisfies the following properties:

- 1.  $C[\mathcal{B}, \mathcal{R}^{\leq \ell, \mathcal{A}}] = \overline{C}[\mathcal{A}, \pi(\mathcal{R})^{\leq \ell, \mathcal{A}}]$ , and
- 2.  $\mathcal{B}$  incurs a miss at time  $\ell$  on  $\mathcal{R}$  if and only if  $\mathcal{A}$  incurs a miss at time  $\ell$  on  $\pi(\mathcal{R})$ .

We prove the proposition by induction on the timestep  $\ell$ . Suppose that the claim holds for  $\ell < n$ : we will show that it holds for  $\ell + 1$ . By construction, the claim holds for  $\ell = j + 1$ ; note that the actions of  $\mathcal{A}$  on  $\pi(\mathcal{R})$  at time j+1 and choice of initial tags guarantee that  $C[\mathcal{B}, \mathcal{R}^{\leq j+1,\mathcal{A}}] = \overline{C}[\mathcal{A}, \pi(\mathcal{R})^{\leq j+1,\mathcal{A}}]$ . We now use case analysis at timestep  $\ell + 1$  on requests  $\mathcal{S}_{\mathcal{R}_i,\mathcal{B}}[\ell + 1]$  for  $i = 1, \ldots, p$  where  $\mathcal{S}_{\mathcal{R}_i,\mathcal{B}}[\ell + 1]$  is the request by  $p_i$  at time  $\ell + 1$  while  $\mathcal{B}$  serves  $\mathcal{R}$ . Similarly,  $\mathcal{S}_{\pi(\mathcal{R}_i),\mathcal{A}}[\ell + 1]$  for  $i = 1, \ldots, p$  is the request by  $p_i$  at time  $\ell + 1$  while  $\mathcal{A}$  serves  $\pi(\mathcal{R})$ .

- **Case 1.** If  $\mathcal{S}_{\mathcal{R}_i,\mathcal{B}}[\ell+1] \neq \sigma_{\text{NLRU}}, \sigma_{\text{LRU}}$ , then
  - $\sigma_{\text{NLRU}}, \sigma_{\text{LRU}} \neq S_{\pi(\mathcal{R}_i),\mathcal{A}}[\ell+1].$  If a request  $S_{\mathcal{R}_i,\mathcal{B}}[\ell+1]$  is a hit for  $\mathcal{B}$ , it is also a hit for  $\mathcal{A}$ , and both  $\mathcal{A}$  and  $\mathcal{B}$  will update the tag of page  $S_{\mathcal{R}_i,\mathcal{B}}[\ell+1]$  to  $\ell+1$  in their corresponding caches. Similarly, if  $S_{\mathcal{R}_i,\mathcal{B}}[\ell+1]$  is a miss for  $\mathcal{B}$ , then by the induction hypothesis about the cache configuration of  $\mathcal{A}, \overline{S_{\mathcal{R}_i,\mathcal{B}}[\ell+1]}$  will also be a miss in  $\mathcal{A}$ . Additionally,  $\mathcal{A}$  and  $\mathcal{B}$  will evict the same page from their cache and update the tag of  $S_{\mathcal{R}_i,\mathcal{B}}[\ell+1]$  to  $\ell+1$ , so the proposition holds for  $\ell+1$ .
- **Case 2.** If  $S_{\mathcal{R}_i,\mathcal{B}}[\ell+1] = \sigma_{\text{LRU}}$ , then  $S_{\pi(\mathcal{R}_i),\mathcal{A}}[\ell+1] = \sigma_{\text{NLRU}}$ . We consider two cases: either  $S_{\mathcal{R}_i,\mathcal{B}}[\ell+1]$  is a hit or miss for  $\mathcal{B}$ . If it was a hit, then by the induction hypothesis  $\sigma_{\text{NLRU}} \in C[\mathcal{A}, \pi(\mathcal{R})^{\leq \ell, \mathcal{A}}]$  and  $S_{\pi(\mathcal{R}_i),\mathcal{A}}[\ell+1]$  is a hit in  $\mathcal{A}$ . After serving request  $S_{\mathcal{R}_i,\mathcal{B}}[\ell+1], \mathcal{B}$  updates the tag of  $\sigma_{\text{LRU}}$  to  $\ell+1$ , and  $\mathcal{A}$  sets the tag of  $\sigma_{\text{NLRU}}$  to  $\ell+1$ , so  $C[\mathcal{B}, \mathcal{R}^{\leq \ell+1,\mathcal{B}}] = \overline{C}[\mathcal{A}, \pi(\mathcal{R})^{\leq \ell+1,\mathcal{A}}]$ . If  $S_{\mathcal{R},\mathcal{B}}[\ell+1]_i$  was a miss for  $\mathcal{B}$ , then from the induction hypothesis  $S_{\pi(\mathcal{R}_i),\mathcal{A}}[\ell+1]$  was not in  $\mathcal{A}$ 's cache at time  $\ell$ . Therefore,  $\mathcal{A}$  and

 $\mathcal{B}$  evict the same page in order to bring in  $\sigma_{\text{LRU}}$ and  $\sigma_{\text{NLRU}}$ , respectively, and update the respective tags to  $\ell + 1$ . Therefore, we maintain the invariant that  $C[\mathcal{B}, \mathcal{R}^{\leq \ell+1, \mathcal{B}}] = \overline{C}[\mathcal{A}, \pi(\mathcal{R})^{\leq \ell+1, \mathcal{A}}].$ 

**Case 3.** If  $\mathcal{S}_{\mathcal{R}_i,\mathcal{B}}[\ell+1] = \sigma_{\text{NLRU}}$ , then  $\mathcal{S}_{\pi(\mathcal{R}_i),\mathcal{A}}[\ell+1] = \sigma_{\text{LRU}}$ . We use a symmetric argument to Case 2.

PROPOSITION B.7. (CASES 2, 3, 4 OF  $\pi(\mathcal{R})$ ) If  $\mathcal{R}^{pre}r_{j+1}\overline{\mathcal{R}^{suf}}$  is not consistent with f, then  $\mathcal{B}(\mathcal{R}) \leq \mathcal{A}(\pi(\mathcal{R})).$ 

*Proof.* We proceed by case analysis on  $\pi(\mathcal{R})$ . From construction of  $\mathcal{B}$ ,  $\mathcal{B}(\mathcal{R}^{\leq j+1,\mathcal{B}}) = \mathcal{A}(\pi(\mathcal{R}^{\leq j+1,\mathcal{B}}))$ . Additionally, from initial choice of tags,  $C[\mathcal{B}, \mathcal{R}^{\leq j+1,\mathcal{B}}] = \overline{C}[\mathcal{A}, \pi(\mathcal{R}^{\leq j+1,\mathcal{A}})]$ . Specifically,

 $C[\mathcal{B}, \mathcal{R}^{\leq j+1,\mathcal{B}}], C[\mathcal{A}, \mathcal{R}^{\leq j+1,\mathcal{B}}]$  have identical page sets, except that the first contains  $\sigma_{\text{NLRU}}$  and the second contains  $\sigma_{\text{LRU}}$ . Since  $\mathcal{R}^{\text{pre}}r_{j+1}\overline{\mathcal{R}}^{\text{suf}}$  is not consistent with f, Lemma 5.1 states that both  $\sigma_{\text{LRU}}$  and  $\sigma_{\text{NLRU}}$  must appear in the suffix  $\mathcal{R}^{\text{suf}}$  and that  $\sigma_{\text{NLRU}}$  must be requested earlier (in time) than  $\sigma_{\text{LRU}}$  in  $\mathcal{R}^{\text{suf}}$ .

- **Case 2 of**  $\pi(\mathcal{R})$ . If  $\mathcal{R}^{\text{pre}}r_{i+1}\overline{\mathcal{R}^{\text{suf}}}$  is not consistent with f and  $\mathcal{B}$  incurs a miss before the first request to  $\sigma_{\text{LRU}}$  in  $\mathcal{R}^{\text{suf}}$ ,  $\pi(\mathcal{R}) = \mathcal{R}$ . Suppose that the first request to  $\sigma_{LRU}$  in  $\mathcal{R}^{suf}$  occurs at timestep t and let  $\ell$   $(j + 1 < \ell < t)$  be the earliest timestep on which  $\mathcal{B}$  incurs a miss before t. Let  $\sigma_{\ell}^{i}$  be the page that caused the miss at time  $\ell$ requested by  $p_i$ :  $\sigma_{\ell}^i$  cannot be  $\sigma_{\text{LRU}}$ . Every request up to time  $\ell$  must have been a hit for  $\mathcal{B}$ , and  $C[\mathcal{B}, \mathcal{R}^{<\ell, \mathcal{B}}] = \overline{C}[\mathcal{A}, \pi(\mathcal{R})^{<\ell, \mathcal{A}}].$  On request  $\sigma_{\ell}^{i}$ ,  $\mathcal{B}$  incurs a miss, evicts  $\sigma_{\text{LRU}}$  (in an LRU-like decision), and brings  $\sigma_{\ell}^{i}$  to the cache, and sets its tag to  $\ell$ . Since  $\sigma_{\ell}^i \notin \{\sigma_{\text{LRU}}, \sigma_{\text{NLRU}}\}, \mathcal{A}$  will also incur a miss in  $\pi(\mathcal{R})$  at time  $\ell$  on  $\sigma_{\ell}^{i}$  and replace  $\sigma_{\rm NLRU}$  with  $\sigma_{\ell}^i$  in a tag-based eviction (and also set the tag of  $\sigma_{\ell}^i$  to  $\ell$ ). Therefore,  $\mathcal{A}$ and  $\mathcal{B}$  have all the same eviction decisions after time  $\ell$  because  $C[\mathcal{B}, \mathcal{R}^{\leq \ell, \mathcal{B}}] = C[\mathcal{A}, \pi(\mathcal{R})^{\leq \ell, \mathcal{A}}]$ , and  $\mathcal{B}(\mathcal{R}) = \mathcal{A}(\pi(\mathcal{R})).$
- **Case 3 of**  $\pi(\mathcal{R})$ **.** Suppose that the first request to  $\sigma_{\text{LRU}}$  in  $\mathcal{R}^{\text{suf}}$  occurs at time t. If  $\mathcal{R}^{\text{pre}}r_{j+1}\overline{\mathcal{R}^{\text{suf}}}$  is not consistent with f,  $\mathcal{B}$  does not incur a miss between times j + 1 and t, and  $\mathcal{R} \in \mathcal{V}_{\sigma_{\text{NLRU}},\mathcal{I}^{f},\mathcal{A},\mathcal{B}}$ ,  $\pi(\mathcal{R}) = \mathcal{V}_{\sigma_{\text{NLRU}},\mathcal{R},\mathcal{A},\mathcal{B}}$ . In this case,  $\mathcal{A}$  also does not incur any misses between times j + 1 and t. On request  $\sigma_{i}^{t} = \sigma_{\text{NLRU}}$ ,  $\mathcal{B}$  hits on  $\sigma_{\text{NLRU}}$  and  $\mathcal{A}$  incurs a miss and makes an LRU-like eviction: specifically, it evicts  $\sigma_{\text{LRU}}$ , replaces it with  $\sigma_{\text{NLRU}}$ ,

and updates its tag to  $t + \tau$  (after it is done fetching). At time  $t + \tau$ , the cache states of  $\mathcal{A}$  and  $\mathcal{B}$  are the same  $(C[\mathcal{B}, \mathcal{R}^{\leq t+\tau, \mathcal{B}}] = C[\mathcal{A}, \mathcal{R}^{\leq t+\tau, \mathcal{A}}])$ . Additionally,  $\mathcal{B}$ ,  $\mathcal{A}$  are tag-based on each request in  $\mathcal{R}^{>t+\tau, \mathcal{B}}, \pi(\mathcal{R})^{>t+\tau, \mathcal{A}}$  (which happen to be the same). Therefore, the actions of  $\mathcal{A}$  and  $\mathcal{B}$  are the same after time  $t + \tau$ , and so  $\mathcal{B}(\mathcal{R}) = \mathcal{A}(\pi(\mathcal{R}))$ because they have the same total time (repeated requests in  $\mathcal{B}$  to match the miss in  $\mathcal{A}$ ).

**Case 4 of**  $\pi(\mathcal{R})$ . If  $\mathcal{R}^{\text{pre}}r_{j+1}\overline{\mathcal{R}^{\text{suf}}}$  is not consistent with  $f, \mathcal{B}$  does not incur a miss before the first request to  $\sigma_{\text{LRU}}$  in  $\mathcal{R}^{\text{suf}}$ , and  $\mathcal{R} \notin \mathcal{V}_{\sigma_{\text{NLRU}},\mathcal{I}^{f},\mathcal{A},\mathcal{B}}$ , then  $\pi(\mathcal{R}) = \mathcal{R}'$ . In this case,  $\mathcal{B}(\mathcal{R}) < \mathcal{A}(\pi(\mathcal{R}))$  by construction of  $\mathcal{R}'$ .